MATEMATIK
Chalmers tekniska högskola och Göteborgs universitet
Lösningsförslag till
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1. Consider
$$\begin{cases} u''(x) + u'(x) + \lambda \arctan(u(x^2)) = 0, & 0 \le x \le 1 \\ u(0) = u(1) = 1, & u \in C^2([0,1]) \end{cases}$$
To remove the inhomogeneous boundary conditions set $v(x) = u(x) - 1$ for $x \in [0,1]$. Here v satisfies
$$(*)\begin{cases} Lv(x) \equiv v''(x) + v'(x) = -\lambda \arctan(1 + v(x^2))), & x \in [0,1] \\ BC : v(0) = v(1) = 0, & v \in C^2([0,1]). \end{cases}$$
Step 1: Calculate the Green's function for L, BC.
$v_1(x) = 1$, $v_2(x) = e^{-x}$ forms a basis for $\mathcal{N}(L)$. We note that
$$\det \begin{bmatrix} v_1(0) & v_1(1) \\ v_2(0) & v_2(1) \end{bmatrix} = \frac{1}{e} - 1 \neq 0,$$
which (using the notation from the lecture notes on ODE) implies that $L_0 : C^2_R([0,1]) \to C([0,1])$ is a bijection.
Set $e(x,t) = a_1(t)v_1(x) + a_2(t)v_2(x)$ where $e(t,t) = 0$ and $e'_x(t,t) = 1$ for all $t \in [0,1]$. We get
$$\begin{cases} a_1(t) + a_2(t)e^{-t} = 0 \\ -a_2(t)e^{-t} = 1 \end{cases} \quad \text{i.e.} \quad \begin{cases} a_1(t) = 1 \\ a_2(t) = -e^t \end{cases}$$
Set $g(x,t) = e(x,t)\theta(x-t) + b_1(t)v_1(x) + b_2(t)v_2(x)$ where $g(0,t) = g(1,t) = 0$ for all $0 < t < 1$. This gives us
$$\begin{cases} b_1(t) + b_2(t) = 0 \\ 1 - e^{t-1} + b_1(t) + b_2(t)e^{-1} = 0 \end{cases} \quad \text{i.e.} \quad \begin{cases} b_1(t) = \frac{e^t - e}{e-1} \\ b_2(t) = -\frac{e^t - e}{e-1} \end{cases}$$
and yields
$$g(x,t) = (1 - e^{t-x})\theta(x-t) + \frac{e^t - e}{e-1} - \frac{e^t - e}{e-1}e^{-x}.$$
Step 2: The BVP $(*)$ is equivalent to the integral equation
$$v(x) = \int_0^1 g(x,t)(-\lambda \arctan(1 + v(t^2)))\, dt.$$
For $v \in C([0,1])$ define
$$T(v)(x) = \int_0^1 g(x,t)(-\lambda \arctan(1 + v(t^2)))\, dt, \quad x \in [0,1].$$
Then T is a mapping from $C([0,1])$ into $C([0,1])$, actually we have $T(v) \in C^2([0,1])$ for $v \in C([0,1])$. Here $(*)$ has a solution if T has a fixed point and the solution is

unique if the fixed point is unique. Assume that $C([0,1])$ is equipped with the norm $\|f\| = \max_{0\leq x\leq 1} |f(x)|$ so that $(C([0,1]), \|\cdot\|)$ becomes a Banach space.

Step 3: Assume that $|\lambda| \leq 1$.
T is a contraction on $C([0,1])$ since:
For $x \in [0,1]$ and $w_1, w_2 \in C([0,1])$ we have

$$|T(w_1)(x) - T(w_2)(x)| = |\int_0^1 g(x,t)(\lambda(\arctan(1+w_2(t^2)) - \arctan(1+w_1(t^2))))\,dt| \leq$$

$$\leq |\lambda| \int_0^1 |g(x,t)|\cdot|\arctan(1+w_2(t^2)) - \arctan(1+w_1(t^2))|\,dt \leq \{\text{mean value theorem}\} \leq$$

$$\leq |\lambda| \int_0^1 |g(x,t)|\,dt \cdot \|w_1 - w_2\|$$

and hence

$$\|T(w_1) - T(w_2)\| \leq |\lambda| \max_{0\leq x\leq 1} \int_0^1 |g(x,t)|\,dt \cdot \|w_1 - w_2\|.$$

Moreover $g(x,t) \leq 0$ for $x, t \in [0,1]$ so

$$\int_0^1 |g(x,t)|\,dt = \int_0^1 -g(x,t)\,dt \equiv j(x),\ x \in [0,1]$$

where $j(x)$ is the solution to $j''(x) + j'(x) = -1$, $j(0) = j(1) = 0$, which gives $j(x) = \frac{e}{e-1}(1-e^{-x}) - x$ and $\max_{0\leq x\leq 1} |j(x)| = \frac{1}{e-1} + 1 + \ln(1-\frac{1}{e}) \leq \frac{1}{e-1} + 1 - \frac{1}{e} = 1 - \frac{1}{e(e-1)} < 1$. Hence T is a contraction. The Banach's fixed point theorem implies that T has a unique fixed point and $(*)$ has a unique solution.

Step 4: λ is an arbitrary real number.
Now T is no longer a contraction and we cannot use Banach's fixed point theorem. Instead we use Schauder's fixed point theorem, which implies that we cannot prove uniqueness (unless we give additional arguments). We note that

$$|\lambda \arctan(1 + v(t^2))| \leq \frac{\pi}{2}|\lambda|$$

for every $t \in [0,1]$ and $v \in C([0,1])$ and

$$|g(x,t)| \leq 2$$

for every $x, t \in [0,1]$. Hence

$$\|T(v)\| \leq \int_0^1 2\cdot\frac{\pi}{2}|\lambda|\,dt = \pi|\lambda|$$

for every $v \in C([0,1])$. Set $S = \{v \in C([0,1]) : \|v\| \leq \pi|\lambda|\}$. Here S is a closed convex subset of $C([0,1])$ and $T(S) \subset S$. It remains to prove that $T(S)$ is relatively compact in $C([0,1])$ and that T is continuous on S. In step 3 we showed that T is continuous (the fact that $|\lambda| \leq 1$ was not used) and applying Arzela-Ascoli theorem we can conclude that $T(S)$ is relatively compact in $C([0,1])$ provided we can establish that

(a) $T(S)$ is bounded in $C([0,1])$, which is obvious from above, and

(b) $T(S)$ is equicontionuous, which follows from $g(x,t)$ being continuous on the compact set $[0,1]\times[0,1]$ and hence uniformly continuous on that set.

It follows that T has a fixed point in S and hence the BVP has a solution, possibly not unique.

2. $(x_n)_{n=1}^{\infty}$ is an ON-sequence in a Hilbert space H and $(c_n)_{n=1}^{\infty}$ is a sequence of complex numbers. We define
$$T(x) = \Sigma_{n=1}^{\infty} c_n\langle x, x_n\rangle x_n, \quad x \in H.$$
For this to be a well-defined mapping $\Sigma_{n=1}^{\infty} c_n\langle x, x_n\rangle x_n$ has to converge for every $x \in H$. Since H is a Hilbert space it is enough to show that the partial sums $(s_N(x))_{N=1}^{\infty}$ form Cauchy sequences in H for all $x \in H$, where
$$s_N(x) = \Sigma_{n=1}^{N} c_n\langle x, x_n\rangle x_n, \quad N = 1,2,3,\dots$$
For $N > M$ we have
$$\|s_N(x) - s_M(x)\|^2 = \|\Sigma_{n=M+1}^{N} c_n\langle x, x_n\rangle x_n\|^2 = \Sigma_{n=M+1}^{N}|c_n|^2|\langle x, x_n\rangle|^2.$$
Hence if $\sup_{n=1,2,3,\dots}|c_n| \equiv C < \infty$ then
$$\|s_N(x) - s_M(x)\|^2 \le C^2\Sigma_{n=M+1}^{N}|\langle x, x_n\rangle|^2 \to 0, \ M,N \to \infty$$
since $\Sigma_{n=1}^{\infty}|\langle x, x_n\rangle|^2 \le \|x\|^2 < \infty$ by Bessel's inequality.
Claim: If $(c_n)_{n=1}^{\infty}$ is unbounded then there exists a $y \in H$ such that $\Sigma_{n=1}^{\infty} c_n\langle y, x_n\rangle x_n$ does not converge.
Assume that $(c_n)_{n=1}^{\infty}$ is unbounded and let $(c_{n_k})_{k=1}^{\infty}$ be a subsequence of $(c_n)_{n=1}^{\infty}$ such that
$$|c_{n_k}| \ge 2^k, \quad k = 1,2,3,\dots.$$
Set $y \in H$ by $y = \Sigma_{k=1}^{\infty} 2^{-k}x_{n_k}$. Clearly $\Sigma_{k=1}^{\infty} 2^{-k}x_{n_k}$ converges in H, $\langle y, x_{n_k}\rangle = 2^{-k}$ for all k and $\langle y, x_n\rangle = 0$ for $n \ne n_k$ for all k. This implies that
$$\|s_N(y) - s_M(y)|^2 = \Sigma_{n=M+1}^{N}|c_n|^2|\langle y, x_n\rangle|^2 \ge$$
$$\ge \#\{n_k : n_k \in [M+1, N] \text{ some } k = 1,2,3,\dots\} \to \infty$$
as $N \to \infty$ for all fixed M. Hence $\Sigma_{n=1}^{\infty} c_n\langle y, x_n\rangle x_n$ does not converge.
We conclude that T is well-defined iff the sequence $(c_n)_{n=1}^{\infty}$ is bounded. Moreover T is a linear operator on H if $(c_n)_{n=1}^{\infty}$ is bounded. To see this fix $x, y \in H$ and complex numbers α, β. Then
$$T(\alpha x+\beta y) = \lim_{N\to\infty}\Sigma_{n=1}^{N} c_n\langle \alpha x+\beta y, x_n\rangle x_n = \lim_{N\to\infty}(\alpha\Sigma_{n=1}^{N} c_n\langle x, x_n\rangle x_n+\beta\Sigma_{n=1}^{N} c_n\langle y, x_n\rangle x_n) =$$
$$= \alpha\lim_{N\to\infty}\Sigma_{n=1}^{N} c_n\langle x, x_n\rangle x_n + \beta\lim_{N\to\infty}\Sigma_{n=1}^{N} c_n\langle y, x_n\rangle x_n = \alpha T(x) + \beta T(y).$$
Finally T is a bounded linear operator on H if $(c_n)_{n=1}^{\infty}$ is bounded. To see this set $C = \sup_{n=1,2,3,\dots}|c_n|$ and note that
$$\|T(x)\|^2 \le C^2\Sigma_{n=1}^{\infty}|\langle x, x_n\rangle|^2 \le C^2\|x\|^2$$

by Bessel's inequality. Hence $\|T\| \leq C$. Actually $\|T\| = \sup_{n=1,2,3,\dots} |c_n|$ which is easily seen.

Next claim is that T is a compact operator iff $\lim_{n\to\infty} c_n = 0$.
Setting $T_N(x) = \Sigma_{n=1}^N c_n\langle x, x_n\rangle x_n,\ \ x \in H$ for $N = 1, 2, 3, \dots$ we see that

(a) T_N is compact for every N since $\dim \mathcal{R}(T_N) = N < \infty$, and

(b) $\|T - T_N\| = \sup_{n=N+1,N+2,N+3,\dots} |c_n| \to 0$ as $N \to \infty$ provided $\lim_{n\to\infty} c_n = 0$, and

(c) $\mathcal{K}(H, H)$ is closed in $\mathcal{B}(H, H)$.

Hence T is a compact operator if $\lim_{n\to\infty} c_n = 0$.
Now assume that $\lim_{n\to\infty} c_n = 0$ does not hold, i.e. there exist a subsequence $(c_{n_k})_{k=1}^\infty$ of $(c_n)_{n=1}^\infty$ such that $\inf_{k=1,2,3,\dots} |c_{n_k}| \equiv c > 0$. Then $(x_{n_k})_{k=1}^\infty$ is a bounded sequence in H but $(T(x_{n_k}))_{k=1}^\infty$ has no converging subsequence since $\|T(x_{n_k}) - T(x_{n_l})\|^2 = |c_{n_k}|^2 + |c_{n_l}|^2 \geq 2c$ for $k \neq l$.
Answer: T is a well-defined bounded linear operator iff the sequence $(c_n)_{n=1}^\infty$ is bounded and T is a compact operator iff $\lim_{n\to\infty} c_n = 0$.

3. For $n = 1, 2, 3, \dots$ define $T_n : C([0, 1]) \to C([0, 1])$ by

$$T_n f(x) = f(x^{1+\frac{1}{n}}),\ x \in [0, 1],$$

where $C([0, 1])$ is equipped with the max-norm.

(a) $T_n f \to f$ in $C([0, 1])$ as $n \to \infty$.
Proof: Fix $f \in C([0, 1])$. Then

$$\|T_n f - f\| = \max_{0\leq x\leq 1} |T_n f(x) - f(x)| = |T_n f(x_n) - f(x_n)|$$

for some $x_n \in [0, 1]$ since $T_n f - f \in C([0, 1])$ and $[0, 1]$ is compact.
Assume that $T_n f \not\to f$ in $C([0, 1])$ as $n \to \infty$. Then there exists an $\epsilon > 0$ such that $\|T_n f - f\| \geq \epsilon$ for infinitely many n. But among these n there exists a converging subsequence $(x_{n_k})_{k=1}^\infty$ of $(x_n)_{n=1}^\infty$. Hence

$$x_{n_k} \to \tilde{x} \text{ as } k \to \infty$$

for some $\tilde{x} \in [0, 1]$, and

$$x_{n_k}^{1+\frac{1}{n_k}} = e^{(1+\frac{1}{n_k})\ln x_{n_k}} \to \tilde{x} \in [0, 1] \text{ as } k \to \infty$$

and

$$\|T_{n_k} f - f\| = |T_{n_k} f(x_{n_k}) - f(x_{n_k})| = |f(x_{n_k}^{1+\frac{1}{n_k}}) - f(x_{n_k})| \geq \epsilon$$

for all k. This gives a contradiction. Hence $T_n f \to f$ in $C([0, 1])$ as $n \to \infty$.

(b) $\|T_n - I\| = 2$ for all n.
Proof: Clearly $\|T_n - I\| \leq \|T_n\| + \|I\| = 2$ for all n. For fixed n set

$$f_n(x) = \begin{cases} 1 & x \in [0, \frac{1}{2}] \\ \text{linear} & x \in [\frac{1}{2}, (\frac{1}{2})^{\frac{n}{n+1}}] \\ -1 & x \in [(\frac{1}{2})^{\frac{n}{n+1}}, 1] \end{cases} \qquad \in C([0, 1]).$$

Then we get $\|f_n\| = 1$ and

$$(T_n - I)f_n((\frac{1}{2})^{\frac{n}{n+1}}) = f_n(\frac{1}{2}) - f_n((\frac{1}{2})^{\frac{n}{n+1}}) = 1 - (-1) = 2.$$

This implies

$$\sup_{\|f\|=1} \|(T_n - I)(f)\| \geq 2.$$

The statement is proved.

4. (a) See textbook

   (b) Consider for example $E = L^2([0,1]) \bigcap C([0,1])$ with the inner product $\langle \cdot, \cdot \rangle_{L^2}$ and let $T : E \to \mathbb{C}$ be defined by

   $$T(f) = \int_0^{\frac{1}{2}} f(t)\, dt.$$

   Clearly T is a linear functional on E with

   $$|T(f)| \leq \frac{1}{\sqrt{2}} \|f\|_{L^2},$$

   using Schwartz' inequality, and so $\|T\| \leq \frac{1}{\sqrt{2}} < \infty$. However if

   $$T(f) = \langle f, g \rangle_{L^2}, \quad f \in L^2$$

   for some $g \in E$ then g must be equal to 1 on $(0, \frac{1}{2})$ and equal to 0 on $(\frac{1}{2}, 1)$ which is impossible.

5. (a) See textbook on Neumann series

   (b) See method of continuity in the lecture notes on spectral theory.
   Set $A_t = I + tS$ for $t \in [0,1]$. We then observe that

      i. $A_0 = I$ is invertible,

      ii. $\|A_t - A_{t'}\| = \|(t - t')S\| = |t - t'|\|S\|$, and

      iii. $\|A_t x\|^2 = \langle (I+tS)(x), (I+tS)(x) \rangle = \|x\|^2 + t^2\|S(x)\|^2 + t\langle S(x), x\rangle + t\langle x, S(x)\rangle \geq \|x\|^2 + t^2\|S(x)\|^2 \geq \|x\|^2$ for $t \in [0,1]$.

   The method of continuity yields that $A_1 = I + S$ is invertible.

6. From the textbook we have that $O \neq T \in \mathcal{B}(H,H)$, where H is a Hilbert space, satisfying $T^2 = T$ is an ortogonal projection iff T is self-adjoint. To prove that the statements a), b) and c) are equivalent we prove the four implications below. The following implication suffices.
a) $\Rightarrow$ b): $T^* = T$ implies $\mathcal{N}(T^*) = \mathcal{N}(T)$ and hence

$$\mathcal{N}(T) = \mathcal{N}(T^*) = \{\text{well} - \text{known}\} = \mathcal{R}(T)^{\perp}$$

follows.
a) $\Rightarrow$ c): Since T is an orthogonal projection on H we have that $T = P_S$ for some closed

subspace of H. Moreover $H = S \bigoplus S^{\perp}$ and $H \ni x = y + z$, where $y \in S$ and $z \in S^{\perp}$, implies $T(x) = y$. This gives

$$\langle T(x), x\rangle = \langle y, x\rangle = \langle y, y\rangle = \|y\|^2 \geq 0$$

and the implication a) $\Rightarrow$ c) follows.
b) $\Rightarrow$ a): $T^2 = T$ implies $\mathcal{N}(I - T) = \mathcal{R}(T)$ and hence $\mathcal{R}(T)$ is a closed subspace of H. Set $S = \mathcal{R}(T)$. By the orthogonal projection theorem $H = S \bigoplus S^{\perp}$. Fix $x \in H$. Then $x = y + z$ where $y \in S$ and $z \in S^{\perp}$. But $S^{\perp} = \mathcal{N}(T)$ by b). Hence $T(x) = T(y)$. But $y \in S$ implies that $y = T(w)$ for some $w \in H$ and so $T(y) = T^2(w) = T(w) = y$. We have $T(x) = y$ and a) is proved.
c) $\Rightarrow$ b): Fix $x \in \mathcal{R}(T)$ and $y \in \mathcal{N}(T)$. This yields

$$0 \leq \langle T(x+y), x+y\rangle = \langle T(x), x+y\rangle.$$

But $x \in \mathcal{R}(T)$ implies $x = T(z)$ for some $z \in H$ and so
$T(x) = T^2(z) = T(z) = x$. Hence

$$\langle x, y\rangle \geq -\|x\|^2.$$

But $y \in \mathcal{N}(T)$ implies $ny \in \mathcal{N}(T)$ for $n = 1, 2, 3, \ldots$ and so

$$\langle x, y\rangle \geq -\frac{1}{n}\|x\|^2.$$

Letting $n \to \infty$ implies

$$\langle x, y\rangle \geq 0.$$

Also $y \in \mathcal{N}(T)$ implies $-y \in \mathcal{N}(T)$ yielding

$$\langle x, -y\rangle \geq 0.$$

Hence $\langle x, y\rangle = 0$ and b) follows.