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1 A Note on Fixed Point Theory

1.1 Introduction

This note contains topics from nonlinear functional analysis. It means that the mappings that appear are not assumed to be linear unless explicitly stated to be so.

Our main problem is to solve equations of the form

$$T(u) = v,$$

where $T : X \to Y$ is a mapping between Banach spaces X and Y. Here $v \in Y$ is given and we look for solutions in X or some subset of X. For linear mappings T we can often find a formula for the inverse operator. The solution has to be uniquely defined in this case. An example of this is boundary value problems

$$\begin{cases} u^{(n)} + a_{n-1}u^{(n-1)} + \ldots + a_1 u' + a_0 u = v, & \text{in } I \\ \text{homogeneous boundary values} & \text{on } \partial I \end{cases}$$

The solutions are obtained as convolutions of the Green's function for the problem with the right hand side v of the differential equation.

However if T is a nonlinear mapping then in general we can not find a formula representing the solution/solutions. This is also the case when $X = Y$. We can no longer prove the existence of a solution just by explicitly writing down the inverse operator, but we have rely on mapping properties of T to prove the existence of a solution. It might be the case that there are several solutions.

In connection with integral equations for instance we have $X = Y$ and the mapping T takes the form

$$T(u) = u + G(u),$$

i.e. T is a perturbation of the identity mapping. The problem can be formulated as

$$u = H(u),$$

where $H(u) = v - G(u)$. Here we suppress the variable v and consider H as a function of u with v as a parameter. The problem to find a solution is then equivalent to find a fixed point of H, i.e. an element $u_0 \in X$ such that

$$u_0 = H(u_0).$$

We recall that if G is linear and small in the sense that the operator norm of G is less than 1 then the mapping T^{-1} is a welldefined bounded linear mapping and can be obtained as a Neumann series (see [5]).

The fixed point results that will be discussed here are of two types. The first type deals with contractions and are referred to as metric fixed point theorems. One example of such a theorem is called Banach's fixed point theorem. The second type deals with compact mappings. Those are called topological fixed point theorems and are more involved. Names associated with such results are Brouwer and Schauder.

First let us consider a simple example. Assume that

$$f : [0,1] \to [0,1]$$

is a continuous function. Then there exists a $x_0 \in [0,1]$ such that $f(x_0) = x_0$. This is a consequence of the theorem saying that every real-valued continuous function attains every intermediary value between any two given values and is based on the fact that

1. $[0,1]$ is a connected closed (i.e. a compact[1] and convex) subset in a Banach space, here $\mathbb{R}$, and that
2. f is a continuous function.

To prove the existence of a fixed point for f we just define the function $g(x) = x - f(x)$ on the interval $[0,1]$ and observe that g is a continuous function satisfying $g(0) \leq 0 \leq g(1)$. We can then conclude that there is a $x_0 \in [0,1]$ such that $g(x_0) = 0$. This example can be considered as the 1-dimensional version of Brouwer fixed point theorem. One feature here is that the method is not constructive, i.e. the position of the fixed point is not given by the method. Nor does the method yield that the fixed point is unique, which indeed is sound since there can be any number of fixed points for f. To get some information on the position of one fixed point we can use the strategy of repeatedly bisecting intervals into pieces as follows: Assume that $g(0) < 0 < g(1)$, since otherwise we already have one fixed point, and consider the subintervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$. If $g(\frac{1}{2}) = 0$ we have one fixed point namely $x_0 = \frac{1}{2}$. If $g(\frac{1}{2}) > 0$ or $g(\frac{1}{2}) < 0$ we can apply the procedure to the the restriction of the function g to the subintervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ respectively. In this way we either find a fixed point as an end point of an interval or we find an infinite set of nested shrinking intervals that all contains a fixed point. For the later case we can for any $\epsilon > 0$ find an interval of length less than ϵ that contains a fixed point. We also note that this argument proves the intermediary value theorem provided we have that $\mathbb{R}$ is a complete normed space, i.e. a Banach space. Compare the argument above with the proof of Baire's theorem.

[1]cf. $g : (0,1) \to (0,1)$ with $g(x) = \frac{x}{2}$.

1.2 Banach's fixed point theorem

First we look at the problem to find a fixed point for a real-valued continuous function $f : \mathbb{R} \to \mathbb{R}$ in the spirit of Banach's fixed point theorem. We then need f to be a contraction meaning that there is a positive real number c less than 1 such that for any pair x, y of points the distance between the images under f of these points is smaller by a factor c than the distance between the points x and y. In formulas this means

$$|f(x) - f(y)| \leq c|x - y|$$

for arbitrary $x, y \in \mathbb{R}$. The conclusion from Banach fixed point theorem is that there is a unique fixed point for f. This can be found by just fixing any element $z \in \mathbb{R}$ and then forming the sequence $(T^n(z))_{n=1}^{\infty}$. T^n denotes the operator obtained by composing T with itself n times, i.e. $T^n = \underbrace{T \circ T \circ \ldots \circ T}_{n \text{ elements}}$. The sequence is converging with the fixed point as the limit point. On the other hand there is no restriction on the domain of f being a convex compact set.

We first state and prove some general observations.

Theorem 1.1. *Let T be a continuous mapping on a Banach space X. Then the following statements hold true:*

1. *If there exist $x, y \in X$ such that*

$$\lim_{n\to\infty} T^n(x) = y$$

   then y is a fixed point for T, i.e. $T(y) = y$.

2. *If $T(X)$ is a compact set in X and for each $\epsilon > 0$ there exists a $x_\epsilon \in X$ such that*

$$\|T(x_\epsilon) - x_\epsilon\| < \epsilon$$

   then T has a fixed point.

Proof. Set $y_n = T^n(x)$, $n = 1, 2, \ldots$. If T is a continuous mapping then

$$T(y) = T(\lim_{n\to\infty} y_n) = \lim_{n\to\infty} T(y_n) = \lim_{n\to\infty} y_{n+1} = y,$$

which proves the first statement.

Assume that the assumptions of 2) are satisfied. Then for $n = 1, 2, \ldots$ there are $x_n \in X$ such that

$$\|T(x_n) - x_n\| < \frac{1}{n}. \tag{1}$$

$T(X)$ is a compact set which implies that there exits a convergent subsequence $(T(x_{n_k}))_{k=1}^{\infty}$ of $(T(x_n))_{n=1}^{\infty}$. Call the limit point x. Then x is a fixed point for T since also the sequence $(x_{n_k})_{k=1}^{\infty}$ converges to x according to (1) and T is continuous. □

We now formulate one of the main theorems.

Theorem 1.2 (Banach's fixed point theorem)**.** *Let T be a contraction on a Banach space X. Then T has a unique fixed point.*

Proof. Fix an arbitrary element $z \in X$ and consider the sequence

$$(T^n(z))_{n=1}^{\infty}.$$

Set $z_n = T^n(z)$ for $n = 1, 2, \ldots$. We note that

$$\|z_n - z_m\| \leq \|z_n - z_{n-1}\| + \ldots + \|z_{m+1} - z_m\| =$$

$$= \|T(z_{n-1}) - T(z_{n-2})\| + \ldots + \|T(z_m) - T(z_{m-1})\| \leq$$

$$\leq c\|z_{n-1} - z_{n-2}\| + \ldots + c\|z_m - z_{m-1}\| \leq \ldots \leq$$

$$\leq (c^{n-1} + c^{n-2} + \ldots c^m)\|z_1 - z\| \leq \frac{c^m}{1-c}\|z_1 - z\|,$$

where we (without loss of generality) have assumed $n > m \geq 1$. This yields $\|z_n - z_m\| \to 0$ as $n, m \to \infty$ and hence $(z_n)_{n=1}^{\infty}$ is a Cauchy sequence. Since X is a Banach space the sequence converges, i.e. there is a $x_0 \in X$ such that $z_n \to x_0$ as $n \to \infty$. Here x_0 is a fixed point for T since

$$\|T(x_0) - x_0\| \leq \|T(x_0) - T(z_n)\| + \|z_{n+1} - x_0\| \leq c\|x_0 - z_n\| + \|z_{n+1} - x_0\|$$

where the LHS is independent of n and the RHS tends to 0 as $n \to \infty$. The uniqueness follows from the contraction property for T. If $x_0 \neq y_0$ both are fixed points of T then we get

$$\|x_0 - y_0\| = \|T(x_0) - T(y_0)\| \leq c\|x_0 - y_0\| < \|x_0 - y_0\|$$

which results in a contradiction. □

From the proof we see that

1. the sequence $(T^n(z))_{n=1}^{\infty}$ converges to the unique fixed point **independently** of the choice of z.
2. for an arbitrary element $x \in X$ we have

   $$\|x - x_0\| \leq \frac{1}{1-c}\|x - T(x)\|,$$

   where x_0 denotes the fixed point of T, since

   $$\|x - x_0\| \leq \|x - T(x)\| + \|T(x) - T(x_0)\| \leq \|x - T(x)\| + c\|x - x_0\|.$$

Banach's fixed point theorem can be generalized in the following way.

Theorem 1.3. *Let T be a mapping on a Banach space X such that T^N is a contraction on X for some positive integer N. Then T has a unique fixed point.*

It is not necessary to assume that T is continuous.

Proof. Banach's fixed point theorem implies that there exists a unique fixed point for T^N. Call this element x_0. Now just note that

$$\|T(x_0) - x_0\| = \|T^N(T(x_0)) - T^N(x_0)\| \leq c\|T(x_0) - x_0\|$$

implies that $T(x_0) = x_0$ since $0 < c < 1$. The uniqueness is clear since a fixed point for T is also a fixed point for T^N. $\square$

Note that the conclusion of the previous theorem remains true if $T : F \to F$, where F is a closed set in the Banach space X, and T^N is a contraction for some positive integer N.

We end this section by two examples.

Example: Let $K(x, y)$ be a continuous real-valued function for $0 \leq y \leq x \leq 1$ and let $v(x)$ be a continuous real function for $0 \leq x \leq 1$. Then there is a unique continuous real function $z(x)$ such that

$$z(x) = v(x) + \int_0^x K(x, y)z(y)\, dy, \quad 0 \leq x \leq 1.$$

To prove this we consider the Banach space $C([0, 1])$ with the sup-norm and define the integral operator $L : C([0, 1]) \to C([0, 1])$ by

$$Lz(x) = \int_0^x K(x, y)z(y)\, dy.$$

Clearly L^n will be an integral operator on $C([0, 1])$ given by a kernel function $K_n(x, y)$. To find this function set $K_1(x, y) = K(x, y)$ and assume that $K_n(x, y)$ is known. Then we obtain

$$(L^{n+1}z)(x) = \int_0^x K(x, t)(L^n z)(t)\, dt = \int_0^x K(x, t) \int_0^t K_n(t, y)z(y)\, dy\, dt =$$

$$= \int_0^x (\int_y^x K(x, t)K_n(t, y)\, dt)z(y)\, dy = \int_0^x K_{n+1}(x, y)z(y)\, dy.$$

Hence

$$K_{n+1}(x, y) = \int_y^x K(x, t)K_n(t, y)\, dt, \quad 0 \leq y \leq x \leq 1.$$

The function $K(x, y)$ is continuous on the closed set $\{(x, y) : 0 \leq y \leq x \leq 1\}$ and so it is bounded, say

$$|K(x, y)| \leq M$$

for all $0 \le y \le x \le 1$. Then again by induction we see that

$$|K_n(x,y)| \le \frac{M^n |x-y|^{n-1}}{(n-1)!}$$

for all $0 \le y \le x \le 1$. Indeed if this holds for n then for $0 \le y \le x \le 1$

$$|K_{n+1}(x,y)| \le \int_y^x M \frac{M^n |t-y|^{n-1}}{(n-1)!} \, dt = \frac{M^{n+1}|x-y|^n}{n!}.$$

Hence if n is sufficiently large we have

$$|K_n(x,y)| \le \frac{1}{2}$$

for $0 \le y \le x \le 1$ and so

$$|(L^n z)(x)| \le \int_0^x |K_n(x,y)| \, |z(y)| \, dy \le \frac{1}{2}\|z\|,$$

i.e.

$$\|L^n\| \le \frac{1}{2}.$$

We now define $T : C([0,1]) \to C([0,1])$ by $Tz = v + Lz$. This gives

$$T^n z = (\Sigma_{k=0}^{n-1} L^k) v + L^n z,$$

which yields that T^n is a contraction on $C([0,1])$. By Theorem 1.3 the mapping T has a unique fixed point.

Example: Let $K(x,y)$ and $f(y,z)$ be continuous real-valued functions for $0 \le x, y \le 1$ and $z \in \mathbb{R}$. Moreover let $v(x)$ be a continuous real function for $0 \le x \le 1$. Assume that

$$|f(y,z_1) - f(y,z_2)| \le N|z_1 - z_2|$$

for all $0 \le y \le 1$ and $z_1, z_2 \in \mathbb{R}$. Our claim is that there exists a unique continuous function $z(x)$ on $0 \le x \le 1$ such that

$$z(x) = v(x) + \int_0^x K(x,y) f(y, z(y)) \, dy.$$

As above we define $L : C([0,1]) \to C([0,1])$ by

$$Lz(x) = \int_0^x K(x,y) f(y, z(y)) \, dy$$

and show that the map $T : C([0,1]) \to C([0,1])$, given by

$$T(z) = v + Lz$$

has a unique fixed point. Here comes a nice trick! For $a > 0$ we introduce a new norm $\| \cdot \|_a$ on $C([0,1])$:

$$\|z\|_a = \int_0^1 e^{-ay} |z(y)| \, dy.$$

Then $\|\cdot\|_a$ is indeed a norm on $C([0,1])$ which is equivalent to the L^1 norm. Set $X_a = (C([0,1]), \|\cdot\|_a)$ and let $\tilde{X_a}$ be the completion of X_a. Clearly $\tilde{X_a}$ is the vector space $L^1([0,1])$ with the norm $\|\cdot\|_a$, and L extends to a map $\tilde{L} : \tilde{X_a} \to \tilde{X_a}$ given by the formula for L. Furthermore with

$$M = \max_{0 \le x,y \le 1} |K(x,y)|$$

we have for $z_1, z_2 \in \tilde{X_a}$

$$\|\tilde{L}z_1 - \tilde{L}z_2\|_a = \int_0^1 e^{-ay} \Big| \int_0^y K(y,t)(f(t,z_1(t)) - f(t,z_2(t)))\, dt \Big|\, dy \le$$

$$\le MN \int_0^1 \int_0^y e^{-ay}|z_1(t) - z_2(t)|\, dt\, dy = MN \int_0^1 \int_t^1 e^{-ay}|z_1(t) - z_2(t)|\, dy\, dt =$$

$$= MN \int_0^1 \frac{e^{-at} - e^{-a}}{a} |z_1(t) - z_2(t)|\, dt \le \frac{MN}{a} \|z_1 - z_2\|_a.$$

This shows that for $a > MN$ the map

$$\tilde{L} : \tilde{X_a} \to \tilde{X_a}$$

is a contraction and so is $\tilde{T} = v + \tilde{L}$. It easily follows that $\tilde{T}$ maps $\tilde{X_a}$ into X_a, so the unique fixed point belongs to $C([0,1])$, and is also the unique fixed point for T.

Another version of the trick above is to equip $C([0,1])$ with the norm

$$|z|_a = \sup_{x \in [0,1]} |e^{-ax} z(x)|$$

with a large enough, which is equivalent to the standard sup-norm on $C([0,1])$. The reader is asked to check that the calculations above go through, i.e. L will be a contraction in $(C([0,1]), |\cdot|_a)$. An advantage here is that we do not need to consider any completion $\tilde{X_a}$.

1.3 Brouwer and Schauder fixed point theorems

Let us formulate Brouwer's fixed point theorem.

Theorem 1.4 (Brouwer's fixed point theorem)**.** *Assume that K is a compact convex subset of $\mathbb{R}^n$ and that $T : K \to K$ is a continuous mapping. Then T has a fixed point in K.*

Observe that it does not follow from Brouwer fixed point theorem that the fixed point is unique. Consider for instance the identity operator on a compact convex set K in $\mathbb{R}^n$ for which every $x \in K$ is a fixed point.

Example 1: Take a street map for Goteborg and place it on the floor of a lecture room at Chalmers, say room MVF31. Then there will be a point on the map that coincides with the corresponding point in Goteborg. This follows from both Banach's fixed point theorem and Brouwer's fixed point theorem, where the former theorem also gives that the point is unique. Prove this to yourself!

Example 2: Let T_α denote the rotation α degrees around the center for a closed disc K of radius 1. Then Brouwer's fixed point theorem gives the existence of a fixed point for T_α (of course it is overkill to use a fixed point theorem to see that) while Banach's fixed point theorem cannot be applied directly[2] since T_α is not a contraction. It is obvious that the center of K is a fixed point but Brouwer's fixed point theorem also tells us that it is not possible to compose the rotation with a continuous deformation of the disc into itself in such a way that the composed mapping has no fixed point.

We note that

- (generalization of Brouwer's fixed point theorem): If there exists a homeomorphism, i.e. a continuous bijection with continuous inverse, between a compact convex set K in $\mathbb{R}^n$ and a set $\tilde{K}$, call the homeomorphism φ, and $\tilde{T} : \tilde{K} \to \tilde{K}$ is a continuous mapping then $\tilde{T}$ has a fixed point. To see this consider the mapping $T = \varphi^{-1} \circ \tilde{T} \circ \varphi$.
  Exercise: Prove that $\tilde{T}$ has a fixed point.
- it is enough to prove Brouwer fixed point theorem in the case $K = \overline{B(0,1)}$, where $B(a,r) = \{x \in \mathbb{R}^n : \|x - a\| < r\}$.

There are many proofs for Brouwer's fixed point theorem, both analytical, topological and also combinatorial. One starting point for a proof could be the following. Assume that $K = \overline{B(0,1)}$ and that T has no fixed point. Define the mapping $A : \overline{B(0,1)} \to \overline{B(0,1)}$ as follows: For every inner point x in $\overline{B(0,1)}$ let $\tilde{x}$ denote the point on the boundary $\partial B(0,1)$ that is the intersection of the ray from $T(x)$ through x and the boundary $\partial B(0,1)$. The ray is always well-defined since T has no fixed point. Now set

$$A(x) = \begin{cases} \tilde{x} & \text{if } x \in B(0,1) \\ x & \text{if } x \in \partial B(0,1) \end{cases}$$

Then A is a continuous mapping from $\overline{B(0,1)}$ into $\partial B(0,1)$ (verify this!) such that $A|_{\partial B(0,1)} = I|_{\partial B(0,1)}$. The challenge to show that T has no fixed point is now reformulated as to show that there is no continuous mapping $A : \overline{B(0,1)} \to \partial B(0,1)$ such that $A|_{\partial B(0,1)} = I|_{\partial B(0,1)}$. The statement that there is no such mapping is deep but never the less intuitively obvious. Consider, for $n = 2$, an elastic membrane fixed on a circular frame. The existence of a mapping A implies that it should be possible

[2] Assume that the disc has its center at the origin in $\mathbb{R}^n$. Apply Banach's fixed point theorem to the operators $T_n = (1 - \frac{1}{n})T,\ n = 1, 2, \ldots$. We obtain a sequence of fixed points x_n to T_n such that

$$\|T(x_n) - x_n\| \leq \frac{1}{n}, \quad n = 1, 2, 3, \ldots.$$

The result follows from Theorem 1.1 above.

to deform the membrane continuously in such a way that it in the end coincides with the frame without being fractured. For fixed $x \in B(0,1)$ the mapping

$$t \mapsto (1-t)x + tA(x),\ t \in [0,1]$$

describes how this point on the membrane is moved from x at $t = 0$ to $A(x) \in \partial B(0,1)$ at $t = 1$, under the deformation. Do not forget that the membrane should be fixed at the frame!!!

A beautiful proof based on Sperner's lemma will be indicated in the Exercises [6].

We present Perron's theorem as an application of Brouwer's fixed point theorem. Schauder's fixed point theorem will be applied in the context of nonlinear differential/integral equations to prove the existence of solutions.

Theorem 1.5 (Perron's theorem)**.** *Let A be a real $n \times n$–matrix with positive entries. Then there exists a positive eigenvalue for the linear mapping given by the matrix A, with an eigenvector with positive entries*

To prove Perron's theorem let K denote the set

$$\{(x_1, \ldots, x_n) : x_i \geq 0 \ \text{ all } i,\ \Sigma_{i=1}^n x_i = 1\}$$

and define $T(x) = Ax/\|Ax\|_{l^1}$ for $x \in K$. Apply Brouwer's fixed point theorem.

In a finite-dimensional normed space compactness is equivalent to closedness and boundedness. This is not the case in an infinite-dimensional normed space. The following example due to Kakutani should be compared to the next fixed point theorem due to Schauder.

Example: Let B denote the closed unit ball in $l^2(\mathbb{Z})$, where $l^2(\mathbb{Z})$ consists of all elements $\mathbb{x} = (\ldots, x_{-1}, x_0, x_1 \ldots)$ such that $\|\mathbb{x}\| = (\Sigma_{n=-\infty}^{\infty} |x_n|^2)^{\frac{1}{2}} < \infty$. It is clear that B is convex and bounded. Let $\mathbb{z}$ be the element in $l^2(\mathbb{Z})$ that satisfies $z_0 = 1$ and $z_n = 0$ for $n \neq 0$ and let S denote the shift operator defined by $(S(\mathbb{x}))_n = x_{n-1}$ for $n \in \mathbb{Z}$. Now set

$$T : l^2(\mathbb{Z}) \to l^2(\mathbb{Z}),$$

where

$$T(\mathbb{x}) = S(\mathbb{x}) + (1 - \|\mathbb{x}\|)\mathbb{z}.$$

For $\mathbb{x} \in B$ we have

$$\|T(\mathbb{x})\| \leq \|S(\mathbb{x})\| + (1 - \|\mathbb{x}\|) = 1,$$

i.e. $T(\mathbb{x}) \in B$. But T has no fixed point in B since

$$(T(\mathbb{x}))_n = x_{n-1},\ n \neq 0$$

and

$$(T(\mathbb{x}))_0 = x_{-1} + (1 - \|\mathbb{x}\|),$$

which implies that $x_0 = x_1 = \ldots = x_n = \ldots$ and $x_{-1} = x_{-2} = \ldots = x_{-n} = \ldots$. This yields a contradiction since $\mathbb{x} \in l^2(\mathbb{Z})$.

From this example we see that a generalization of Brouwer's fixed point theorem to infinite-dimensional spaces should have the assumption that $T(K)$ is a compact set. We next formulate two versions of Schauder's fixed point theorem.

Theorem 1.6 (Schauder's fixed point theorem)**.** *Assume that K is a convex compact set in a Banach space X and that $T : K \to K$ is a continuous mapping. Then T has a fixed point.*

For applications the following generalization proves to be useful.

Theorem 1.7 (generalization of Schauder's fixed point theorem)**.** *Let F be a closed convex set in a Banach space X and assume that $T : F \to F$ is a continuous mapping such that $T(F)$ is a relatively compact subset of F. Then T has a fixed point.*

We recall that a set $K_1 \subset X$ is compact[3] if every sequence in K_1 has a convergent subsequence in K_1. Moreover we say that $K_2 \subset X$ is relatively compact if every sequence in K_2 has a subsequence that converges in X. The limit element of the converging sequence belongs to $\overline{K_2}$. The set K_2 being relatively compact implies that $\overline{K_2}$ is a compact set. Also an arbitrary subset of a compact set is relatively compact.

To prove Schauder's fixed point theorem we will make use of some new concepts and facts for compact sets. We say that the convex hull of a set F, denoted by $\operatorname{co} F$, is the set defined by

$$\bigcap_{F \subset H, H \text{ convex}} H.$$

By a convex combination of the elements $x_1, x_2, \ldots, x_n$ we mean a linear combination $\Sigma_{i=1}^n \lambda_i x_n$, where all $\lambda_i \geq 0$ and $\Sigma_{i=1}^n \lambda_i = 1$. An ϵ–net is a subset F_ϵ of F with the property that for each $x \in F$ there exists a $y \in F_\epsilon$ such that $\|x - y\| < \epsilon$.

Proposition 1.1. *The following statements are true:*

1. *A set F is relatively compact iff for each $\epsilon > 0$ there exists a finite ϵ–net.*
2. *A set K is compact iff it is closed and for every $\epsilon > 0$ there exists a finite ϵ–net.*
3. *The set $\operatorname{co} F$ is the same as the set of all convex combination of finitely many elements in F.*
4. *K compact set implies that $\overline{\operatorname{co} K}$ is compact.*

[3]This definition of compactness and relative compactness is sometimes referred to as sequential compactness and sequential relatively compactness in the literature. The words compactness and relatively compactness are then reserved to mean the following: A set K in a normed space is called compact if for each open cover of K there is a finite subcover. An open cover of K is a collection of open sets O_λ, $\lambda \in \Lambda$, whose union contains K as a subset. A finite subcover is a finite subset of $\{O_\lambda\}_{\lambda \in \Lambda}$ whose union also contains the set K. It can be shown that for metric spaces X the notions sequentially compact and compact are equivalent.

The proof is left as an exercise.

Proof. (of the Schauder theorems) The second Schauder theorem is a consequence of the first one. To see this assume that the hypothesis of the second theorem are satisfied. It then follows that the closed hull $\overline{R}$ of $R = T(F)$ is compact and so also $\overline{\text{co}\,\overline{R}}$. Set $K = \overline{\text{co}\,\overline{R}}$. We see that $K \subset F$ since F is closed and convex. Moreover $T : K \to K$ is continuous. Hence the second theorem follows from the first theorem.

It remains to prove the first theorem. This will be done by approximating the compact set K by compact sets K_n, $n = 1, 2, \ldots$ in finite-dimensional spaces and approximating the mapping T by continuous mappings $T_n : K_n \to K_n$, where the approximation becomes better and better for larger n. Brouwer's fixed point theorem gives a sequence of fixed points (x_n) for the sequence (T_n), from which a converging subsequence of points (x_{n_k}) can be extracted. The limit element of this sequence will be a fixed point for T.

For every positive integer n we define mappings P_n, called Schauder projections, as follows: The compactness of K implies that there are finitely many elements $x_1, \ldots, x_k \in K$ such that

$$K \subset \bigcup_{i=1}^{k} B(x_i, \frac{1}{n}).$$

Set

$$f_i(x) = \max(0, \frac{1}{n} - \|x - x_i\|), \ i = 1, \ldots, k.$$

For every $x \in K$ there exists an i such that $f_i(x) > 0$. This implies that $\Sigma_{i=1}^{k} f_i(x) > 0$ for all $x \in K$. Now set $K_n = \overline{\text{co}\{x_1, \ldots, x_k\}}$ and

$$P_n(x) = \frac{\Sigma_{i=1}^{k} f_i(x) x_i}{\Sigma_{i=1}^{k} f_i(x)}, \quad x \in K.$$

Finally we define $T_n = P_n T|_{K_n}$. We can now apply Brouwer's theorem to every mapping

$$T_n : K_n \to K_n, \quad n = 1, 2, \ldots$$

This yields a sequence of fixed points $\tilde{x}_n$ for T_n, i.e.

$$P_n T(\tilde{x}_n) = \tilde{x}_n,$$

and hence we get

$$\|T(\tilde{x}_n) - \tilde{x}_n\| < \frac{1}{n}.$$

Schauder's theorem now follows from Theorem 1.1. □

1.4 Continuity and applications

To apply the fixed point theorems above some results for continuous functions will often be used.

Theorem 1.8. *Assume that T is a continuous mapping between two Banach spaces X and Y. Then the following statements are true:*

1. *If K is a compact set in X then $T(K)$ is a compact set in Y.*
2. *If $Y = \mathbb{R}$ then T attains its maximum and its minimum on every compact set K in X, i.e. there are $x_0, x_1 \in K$ such that*
$$\sup_{x \in K} f(x) = T(x_0) = \max_{x \in K} T(x)$$
and
$$\inf_{x \in K} T(x) = T(x_1) = \min_{x \in K} T(x).$$
3. *T is uniformly continuous on every compact set in X.*

The different notions of continuity that will be used are the following:
Let $T : X \to Y$ be a mapping between two Banach spaces. Then T is called

continuous if for each $x \in X$ and each $\epsilon > 0$ there exists a $\delta = \delta(x, \epsilon) > 0$ such that for every $y \in X$
$$\|y - x\|_X < \delta \Rightarrow \|T(y) - T(x)\|_Y < \epsilon.$$

uniformly continuous on A, where $A \subset X$, if for each $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that for every $x, y \in A$ we have
$$\|y - x\|_X < \delta \Rightarrow \|T(y) - T(x)\|_Y < \epsilon.$$

If $T_\lambda : X \to Y$, $\lambda \in \Lambda$ is a set of mappings (finitely many or infinitely many) between two Banach spaces then these are called

equicontinuous on A, where $A \subset X$, if for each $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that for every pair of elements $x, y \in A$ and every $\lambda \in \Lambda$ we have
$$\|y - x\|_X < \delta \Rightarrow \|T_\lambda(y) - T_\lambda(x)\|_Y < \epsilon.$$

Proof. (of Theorem 1.8) To prove statement 1) let $T : X \to Y$ be a continuous mapping and K a compact set in X. Pick an arbitrary sequence $(y_n) \subset T(K)$. Then there exists a sequence (x_n) in K such that $T(x_n) = y_n$ for all n. The sequence (x_n) might not be uniquely determined since T is not assumed to be injective. But since K is a compact set there exists a convergent subsequence (x_{n_k}) of (x_n) in K, i.e. there is an element $x \in K$ such that $x_{n_k} \to x$ as $k \to \infty$. Moreover since T is continuous we have
$$x_{n_k} \to x \Rightarrow y_{n_k} = T(x_{n_k}) \to T(x) \in T(K).$$
This proves 1).

The proof of statement 2) is left as an exercise.

To prove statement 3) assume that K is a compact set of X and that $T : X \to Y$ is continuous. Moreover assume that T is not uniformly continuous on K. Then there exists an $\epsilon > 0$ such that for all positive integers n there are points $x_n, y_n \in K$ such that

$$\|y_n - x_n\|_X < \frac{1}{n} \tag{2}$$

and

$$\|T(y_n) - T(x_n)\|_Y \geq \epsilon. \tag{3}$$

But K is a compact set and so there exists a convergent subsequence (x_{n_k}) of (x_n), i.e. for some $x \in K$ we have $x_{n_k} \to x$. From (2) it follows that $y_{n_k} \to x$ since we have

$$\|y_{n_k} - x\|_X \leq \|y_{n_k} - x_{n_k}\|_X + \|x_{n_k} - x\|_X.$$

Moreover T is continuous and so $T(x_{n_k}) \to T(x)$ and $T(y_{n_k}) \to T(x)$. This gives a contradiction of (3). The statement 3) is proved. □

The Banach spaces that will be used in applications are $C(A)$ and $L^p(A)$, $1 \leq p < \infty$. Here A stands for different subsets of $\mathbb{R}^n$ for $n \geq 1$. Of course the norms should be the proper ones e.g. the sup-norm should be used for $C(A)$. We tacitly understand that the proper norm is used unless something else is stated. In the context of Schauder's fixed point theorem it is important to be able to conclude whether or not a subset of $C(A)$ or $L^p(A)$ is compact. Our next result answers that question for the case $C(A)$.

Theorem 1.9 (Arzela-Ascoli theorem)**.** *Assume that K is a compact set in $\mathbb{R}^n$, $n \geq 1$ (e.g. $K = [a, b] \subset \mathbb{R}$). Then a set $S \subset C(K)$ is relatively compact in $C(K)$ iff the functions in S are uniformly bounded and equicontinuous on K.*

To say that the functions in S are uniformly bounded means that there exists a $M > 0$ such that

$$\|f\| = \sup_{x \in K} |f(x)| \leq M \quad \text{all} \quad f \in S.$$

To say that the functions in S are equicontinuous on K means that for every $\epsilon > 0$ there exists an $\delta > 0$ such that for every $x, y \in K$ and every $f \in S$ we have

$$|x - y| < \delta \quad \Rightarrow \quad |f(x) - f(y)| < \epsilon.$$

The Arzela-Ascoli theorem can be generalized to the whole of $\mathbb{R}^n$ if we assume that the functions uniformly tends to 0 at infinity i.e. as $|x| \to \infty$.

Next we formulate a criteria for compactness for sets of L^p-functions.

Theorem 1.10 (Riesz, Kolmogorov)**.** *Assume that $1 \leq p < \infty$ and that $S \subset L^p(\mathbb{R}^n)$. Then S is relatively compact in $L^p(\mathbb{R}^n)$ iff the following conditions are satisfied:*

1. *S is a bounded set in $L^p(\mathbb{R}^n)$, i.e. there exists a $M > 0$ such that $\|f\|_{L^p} \leq M$ for all $f \in S$,*

2. $\lim_{x\to 0}\int_{\mathbb{R}^n}|f(y+x)-f(y)|^p\,dy = 0$ *uniformly in* S, *i.e. for every* $\epsilon > 0$ *there exists a* $\delta > 0$ *such that*

$$|x| < \delta \text{ och } f \in S \Rightarrow \|f(\cdot + x) - f(\cdot)\| \equiv (\int_{\mathbb{R}^n} |f(y+x)-f(y)|^p\,dy)^{1/p} < \epsilon,$$

3. $\lim_{R\to\infty}\|f\|_{L^p(\mathbb{R}^n\setminus B(0,R))} = (\int_{|x|>R}|f(x)|^p\,dx)^{1/p} = 0$ *uniformly in* S, *i.e. for every* $\epsilon > 0$ *there exists a* $\omega > 0$ *such that*

$$R > \omega \text{ och } f \in S \Rightarrow (\int_{|x|>R} |f(x)|^p\,dx)^{1/p} < \epsilon.$$

The above results can be found in most textbooks on functional analysis.

We are now ready to apply Schauder's theorem. Note the difference between Schauder's theorem and Banach's theorem, namely to apply Banach's theorem we have to show that a mapping is "sufficiently small", while to apply Schauder's theorem we have to prove that a mapping is compact. This means, in the $C(A)$ or L^p case, that we have to show that the image set for the mapping consists of more "regular" functions.

Example (an integral equation of Hammerstein-type): Assume that $K(x,y)$ is a continuous function for $0 \le x, y \le 1$ and that $f(y,z)$ is a bounded continuous function for $0 \le y \le 1$ and $z \in \mathbb{R}$. Then the equation

$$z(x) = \int_0^1 K(x,y)f(y,z(y))\,dy$$

has a continuous solution $z(x)$.

We want to prove that $T(z)$, $z \in C([0,1])$, has a fixed point where

$$(T(z))(x) = \int_0^1 K(x,y)f(y,z(y))\,dy.$$

To show this we will apply the generalization of Schauder's fixed point theorem. We will choose a closed convex subset $S \subset C([0,1])$ such that the mapping $T : S \to C([0,1])$ is continuous and such that the image set $T(S)$ is relatively compact in $C([0,1])$.

First we observe that T maps continuous functions to continuous functions, i.e. that we have

$$T(C([0,1])) \subset C([0,1]).$$

This can be seen as follows: From the hypothesis there exists a $B > 0$ such that

$$|f(y,z)| \le B \quad \text{if} \quad (y,z) \in [0,1]\times\mathbb{R}.$$

Moreover $K(x,y)$ is continuous on the compact set $[0,1]\times[0,1]$ and hence K is uniformly continuous on $[0,1]\times[0,1]$. Fix an $\epsilon > 0$. Then there exists a $\delta > 0$ such that

$$|K(x,y) - K(\tilde{x},\tilde{y})| < \frac{\epsilon}{B} \quad \text{if} \quad |(x,y) - (\tilde{x},\tilde{y})| < \delta.$$

Consequently for any $z \in C([0,1])$ we have

$$|(T(z))(x) - (T(z))(\tilde{x})| = |\int_0^1 (K(x,y) - K(\tilde{x},y))f(y,z(y))\,dy| \leq$$

$$\leq \int_0^1 |K(x,y) - K(\tilde{x},y)||f(y,z(y))|\,dy \leq B\int_0^1 |K(x,y) - K(\tilde{x},y)|\,dy < \epsilon$$

provided $|x - \tilde{x}| < \delta$. This means that $T(z) \in C([0,1])$.

A natural choice for the closed convex set S is

$$S = \{z \in C([0,1]) : \|z\| \leq D\},$$

where $D > 0$ is a constant that should be chosen such that $T(S) \subset S$. We note that since K is continuous on the compact set $[0,1] \times [0,1]$ there exists an $A > 0$ such that

$$|K(x,y)| \leq A \quad \text{if} \quad (x,y) \in [0,1] \times [0,1].$$

This implies that

$$|(T(z))(x)| = |\int_0^1 K(x,y)f(y,z(y))\,dy| \leq \int_0^1 |K(x,y)||f(y,z(y))|\,dy \leq AB$$

for $z \in C([0,1])$. Hence we get

$$\|T(z)\| \leq D$$

provided we choose $D \geq AB$. Set $D = AB$. With this choice for S we get

$$T(S) \subset S.$$

To apply Schauder's theorem we have to show that $T(S)$ is relatively compact in $C([0,1])$ and that T is continuous on S. The relatively compactness is consequence of Arzela-Ascoli theorem once we have shown that $T(S)$ is uniformly bounded and equicontinuous on S.

We have above verified that $T(C([0,1]))$ is uniformly bounded and equicontinuous on S. It remains to prove that $T : S \to T(S)$ is continuous. From the definition of S it follows that $|z(x)| \leq D$ for all $x \in [0,1]$. The continuity of $f(y,z)$ on the compact set $[0,1] \times [-D,D]$ implies that f is uniformly continuous on $[0,1] \times [-D,D]$. Fix an arbitrary $\epsilon > 0$. Then there exists a $\delta > 0$ such that

$$|f(y,z) - f(\tilde{y},\tilde{z})| < \frac{\epsilon}{A} \quad \text{if} \quad |(y,z) - (\tilde{y},\tilde{z})| < \delta.$$

Hence for arbitrary $z_1, z_2 \in S$ with $\|z_1 - z_2\| < \delta$ we have

$$\|T(z_1) - T(z_2)\| = \sup_{x\in[0,1]} |\int_0^1 K(x,y)(f(y,z_1(y)) - f(y,z_2(y)))\,dy| \leq$$

$$\leq \sup_{x\in[0,1]} \int_0^1 |K(x,y)||(f(y,z_1(y)) - f(y,z_2(y)))|\,dy \leq$$

$$\leq A\int_0^1 |(f(y,z_1(y)) - f(y,z_2(y)))|\,dy < \epsilon.$$

Now we have shown that T is continuous on S. Schauder's fixed point theorem implies that the equation $z = T(z)$ has at least one solution.

1.5 Some more fixed point theorems

We conclude this note with some additional fixed point theorems. The first one, Schaefer's fixed point theorem, is a version of Schauder's theorem. Sometimes it is called the Leray-Schauder principle and is an example of the mathematical principle saying "apriori estimates implies existence". The second one, Krasnoselskii's fixed point theorem, is a mix of Banach's and Schauder's fixed point theorems.

Theorem 1.11 (Schaefer's fixed point theorem)**.** *Assume that X is a Banach space and that $T : X \to X$ is a continuous compact*[4] *mapping. Moreover assume that the set*

$$\bigcup_{0\leq\lambda\leq 1} \{x \in X : x = \lambda T(x)\}$$

is bounded. Then T has a fixed point.

Proof. Assume that the mapping T satisfies the hypothesis in the theorem. Pick a $R > 0$ such that

$$x = \lambda T(x) \;\text{ and }\; 0 \leq \lambda \leq 1$$

implies that

$$\|x\| < R.$$

Define the mapping $\tilde{T} : X \to X$ as follows:

$$\tilde{T}(x) = \begin{cases} T(x) & \text{if } \|T(x)\| \leq R \\ \frac{R}{\|T(x)\|}T(x) & \text{if } \|T(x)\| > R \end{cases}$$

This implies that $\tilde{T} : X \to X$ is a compact operator. To show this take a bounded sequence $(x_n)_{n=1}^\infty$ in X. Then there exists a subsequence $(x_{n_k})_{k=1}^\infty$ such that $\|T(x_{n_k})\| < R$ for all k or $\|T(x_{n_k})\| \geq R$ for all k. In the first case $(\tilde{T}(x_{n_k}))_{k=1}^\infty$ has a convergent subsequence since $\tilde{T}(x_{n_k}) = T(x_{n_k})$ and T is a compact mapping. In the second case we get that $(T(x_{n_k}))_{k=1}^\infty$ has a convergent subsequence, denote it by $(T(x_l))_{l=1}^\infty$ for convenience. But then it follows that also $(\|T(x_l)\|)_{l=1}^\infty$ converges, where also $\|T(x_l)\| \geq R$ for all l. Hence we have $\tilde{T}(x_l) = \frac{R}{\|T(x_l)\|}T(x_l)$.

Set

$$K = \overline{\operatorname{co}\overline{\tilde{T}(B(0,R))}}.$$

Here K is convex (it is the closed convex hull of a set), compact (the convex hull of a compact set is compact and $\tilde{T}$ is a compact mapping) subset of X such that

$$\tilde{T} : K \to K.$$

Schauder's fixed point theorem implies that $\tilde{T}$ has a fixed point $x_0 \in K$. But x_0 is a fixed point for T if $\|T(x_0)\| \leq R$. Assume that $\|T(x_0)\| > R$. This yields a

[4] T is a compact mapping if $(T(x_n))_{n=1}^\infty$ has a convergent subsequence for every bounded sequence $(x_n)_{n=1}^\infty$ in X. Usually by a compact (or completely continuous) mapping one means a continuous mapping with the property above. For linear mappings the continuity follows from this property but it is not true in general for nonlinear mappings.

contradiction since $x_0 = \tilde{T}(x_0) = \lambda T(x_0)$, where $\lambda = \frac{R}{\|T(x_0)\|} \in (0,1)$, since according to the hypothesis of the theorem it should follow that $\|T(x_0)\| = \|x_0\| < R$. This proves the theorem. $\square$

Note that to apply Schaefer's theorem we do not need to prove that a certain set is convex or compact. The problem is reformulated as to show a certain a priori estimate for the operator T.

Theorem 1.12 (Krasnoselskii's fixed point theorem)**.** *Assume that F is a closed bounded convex subset of a Banach space X. Furthermore assume that T_1 and T_2 are mappings from F into X such that*

1. *$T_1(x) + T_2(y) \in F$ for all $x, y \in F$,*
2. *T_1 is a contraction,*
3. *T_2 is continuous and compact.*

Then $T_1 + T_2$ has a fixed point in F.

Proof. Assume that the mappings T_1, T_2 satisfies the hypothesis of the theorem. In particular there exists a $c \in (0,1)$ such that

$$\|T_1(x) - T_1(y)\| \leq c\|x - y\|, \quad x, y \in F.$$

This yields

$$\|(I - T_1)(x) - (I - T_1)(z)\| \geq \|x - z\| - \|T_1(x) - T_1(z)\| \geq (1 - c)\|x - z\|$$

and

$$\|(I - T_1)(x) - (I - T_1)(z)\| \leq \|x - z\| + \|T_1(x) - T_1(z)\| \leq (1 + c)\|x - z\|.$$

Consequently $I - T_1 : F \to (I - T_1)(F)$ is a homeomorphism, and $(I - T_1)^{-1}$ exists as a continuous mapping from $(I - T_1)(F)$. Furthermore we note that for each $y \in F$ the equation

$$x = T_1(x) + T_2(y)$$

has a unique solution $x \in F$ according to Banach's fixed point theorem. From this we conclude that $T_2(y) \in (I - T_1)(F)$ for every $y \in F$ and also that $(I - T_1)^{-1}T_2 : F \to F$ is a well-defined continuous mapping. Since T_2 is a compact mapping it follows that $(I - T_1)^{-1}T_2 : F \to F$ is a compact mapping. Finally the generalization of Schauder's fixed point theorem yields the conclusion of the theorem. $\square$

We recommend anyone interested in fixed point theorems to browse through the books [7] and [3] where additional results and many more references can be found.

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