TMA401/MMA400 Functional Analysis 2012/2013

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Homework assignment 1

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Problem 1 : Let $C^1([0,1])$ be the vector space of all continuously differentiable functions $f : [0,1] \to \mathbb{R}$. Show that

1. $C^1([0,1])$ with the norm $\|f\| + \|f'\|$ is a Banach space,
2. $C^1([0,1])$ with the norm $\|f\|$ is not a Banach space.

Here $\|f\|$ denotes $\max_{x\in[0,1]} |f(x)|$.

Problem 2 : Let $1 < p < \infty$ and let the subspace $A \subset l^p$ be defined by

$$A = \{\mathbb{x} = (x_1, x_2, \ldots, x_n, \ldots) \in l^p : \mathbb{x} \in l^1 \text{ and } \Sigma_{n=1}^{\infty} x_n = 0\}.$$

Show that

$$\overline{A}^{\|\cdot\|_{l^p}} = l^p.$$

Problem 3 : Let $T : C([0,1]) \to \mathbb{R}$ be defined by $Tf = f(1)$. Show that

1. T is continuous if $C([0,1])$ equipped with the norm $\|f\| = \max_{x\in[0,1]} |f(x)|$,
2. T is not continuous if $C([0,1])$ equipped with the norm $\|f\|_p = (\int_0^1 |f(t)|^p \, dt)^{\frac{1}{p}}$, where $1 \le p < \infty$.

Problem 4: Let E denote the real vector space $C([0,1])$ equipped with the norm $\|f\| = \max_{x\in[0,1]} |f(x)|$ and let $T : E \to E$ be a linear mapping. Assume that $Tf(x) \ge 0$ for all $x \in [0,1]$ provided $f(x) \ge 0$ for all $x \in [0,1]$. Show that

1. T is continuous
2. $\|T\| = \sup_{x\in[0,1]} T\mathbf{1}(x)$ where $\mathbf{1}$ denotes the constant function taking the value 1.
3. Let $U : E \to E$ be defined by $Uf(x) = \int_0^1 e^{xt} f(t) \, dt$ and the sequence $U_n : E \to E$, $n = 1, 2, 3, \ldots$ be defined by

   $$U_n f(x) = \int_0^1 \Sigma_{k=0}^{n} \frac{(tx)^k}{k!} f(t) \, dt.$$

   Prove that $\|U_n - U\| \to 0$.