TMA401/MMA400 Functional Analysis 2012/2013
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# Homework assignment 2

Deadline 2012-10-08

**Problem 1:** A mapping $T : \mathbb{R} \to \mathbb{R}$ satisfies a Lipschitz-condition with constant $k$ if

$$|T(x) - T(y)| \leq k|x - y|$$

for all $x, y \in \mathbb{R}$.

1. Is $T$ a contraction?
2. If $T$ is a $C^1$–function with bounded derivative, show that $T$ satisfies a Lipschitz-condition.
3. If $T$ satisfies a Lipschitz-condition, is $T$ then a $C^1$-function with bounded derivative?
4. Assume that $|T(x) - T(y)| \leq k|x - y|^\alpha$ for all $x, y \in \mathbb{R}$ for some $\alpha > 1$. Show that $T$ is a constant.

**Problem 2:** Consider the equation[1]

$$3u(x) = x + (u(x))^2 + \int_0^1 |x - u(y)|^{1/2}\, dy.$$

Show that it has a continuous solution $u$ satisfying $0 \leq u(x) \leq 1$ for $0 \leq x \leq 1$.

**Problem 3:** Find an ON-basis for the subspace $\mathrm{Span}\{1 + x, 1 - x\}$ in $L^2([0, 1])$.

**Problem 4:** Let $(e_n)_{n=1}^\infty$ be an ON-basis for a Hilbert space $H$. Assume that $(f_n)_{n=1}^\infty$ is an ON-sequence in $H$ that satisfies

$$\Sigma_{n=1}^\infty \|e_n - f_n\|^2 < 1. \tag{1}$$

Show that $(f_n)_{n=1}^\infty$ is an ON-basis for $H$. What can be said if (1) is replaced by

$$\Sigma_{n=1}^\infty \|e_n - f_n\|^2 < \infty.$$

[1] Hint: See Krasnoselskii's fixed point theorem in the lecture notes on fixed point theorems.