TMA401/MMA400 Functional Analysis 2013/2014

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Homework assignment 1

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Problem 1 : Let $E = C([0,1])$. Show that

1. if a_k, $k = 1, \ldots, n$ are n distinct points in $[0,1]$ then the functions
$$x \mapsto |x - a_k|, \quad k = 1, \ldots, n$$
are linearly independent on E,
2. the function
$$(x, y) \mapsto |x - y|$$
on $[0,1] \times [0,1]$ cannot be written as a finite sum
$$\Sigma_{i=1}^{n} v_i(x) w_i(y),$$
where $v_i, w_i \in E$, $i = 1, \ldots, n$.

Problem 2 : Find a sequence $x = (x_1, x_2, \ldots)$ with $x_n \to 0$ as $n \to \infty$ that is not in any l^p for $1 \leq p < \infty$. Find a sequence $x = (x_1, x_2, \ldots)$ which is in l^p for all $p > 1$ but not in l^1. Give an example of a subspace in l^2 that is not closed.

Problem 3 : Let $C^1([0,1])$ be the vector space of all continuously differentiable functions $f : [0,1] \to \mathbb{R}$. Show that

1. $C^1([0,1])$ with the norm $\|f\| + \|f'\|$ is a Banach space,
2. $C^1([0,1])$ with the norm $\|f\|$ is not a Banach space.

Here $\|f\|$ denotes $\max_{x \in [0,1]} |f(x)|$.

Problem 4: Let E denote the real vector space $C([0,1])$ equipped with the norm $\|f\| = \max_{x \in [0,1]} |f(x)|$ and let $T : E \to E$ be a linear mapping. Assume that $Tf(x) \geq 0$ for all $x \in [0,1]$ provided $f(x) \geq 0$ for all $x \in [0,1]$. Show that

1. T is continuous
2. $\|T\| = \sup_{x \in [0,1]} T\mathbf{1}(x)$ where $\mathbf{1}$ denotes the constant function taking the value 1.
3. Let $U : E \to E$ be defined by $Uf(x) = \int_0^1 e^{xt} f(t)\, dt$ and the sequence $U_n : E \to E$, $n = 1, 2, 3, \ldots$ be defined by
$$U_n f(x) = \int_0^1 \Sigma_{k=0}^{n} \frac{(tx)^k}{k!} f(t)\, dt.$$
Prove that $\|U_n - U\| \to 0$.