TMA401/MMA400 Functional Analysis 2017/2018

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Homework assignment 1

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Problem 1 : Let $(E, \|\cdot\|)$ be a normed space with a subspace Y. Show that $\overline{Y}$ is a subspace of E.

Problem 2 : Let $l^\infty = \{\mathbb{x} \in \mathbb{R}^\infty : \sup_n |x_n| < \infty\}$ and $c = \{\mathbb{x} \in l^\infty : \lim_{n\to\infty} x_n \text{ exists}\}$ be real vector spaces with the addition and multiplication as in $\mathbb{R}^\infty$. Let both l^∞ and c be equipped with the norm $\|\mathbb{x}\|_\infty = \sup_n |x_n|$. Show that

1. l^∞ and c are Banach spaces
2. $\|\mathbb{x}\|_\infty = \lim_{n\to\infty}(\lim_{p\to\infty}(\Sigma_{k=1}^n |x_n|^p)^{\frac{1}{p}})$ for $\mathbb{x} \in l^\infty$.

Problem 3 : Let $(E, \|\cdot\|)$ be a normed space. Let $B(a; r)$ denote the open ball in E around $a \in E$ with radius $r > 0$. Prove that

1. $\overline{B(a;r)} = \{x \in E : \|x - a\| \leq r\}$ for each $a \in E$ and $r > 0$,
2. if $B(a; r) \subset B(b; s)$ then $r \leq s$ and $\|a - b\| \leq s - r$,
3. and, if $(E, \|\cdot\|)$ is a Banach space, then every nested sequence of nonempty closed balls (i.e. closures of open balls) has nonempty intersection.

Problem 4 : Let $C^1([0, 1])$ be the vector space of all continuously differentiable functions $f : [0, 1] \to \mathbb{R}$. Show that

1. $C^1([0, 1])$ with the norm $\|f\| + \|f'\|$ is a Banach space,
2. $C^1([0, 1])$ with the norm $\|f\|$ is not a Banach space.

Here $\|f\|$ denotes $\max_{x\in[0,1]} |f(x)|$.