TMA401/MMA400 Functional Analysis 2017/2018

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Homework assignment 2

Deadline 2017-09-28

Problem 1: Suppose $L \in (0, \sqrt{(\sqrt{5}-1)/2})$. Show that the equation

$$f(x) - \int_0^L \sqrt{1+(x-y)^2} \cos(f(y))\, dy = \sin(e^x), \quad x \in [0, L]$$

has a unique solution $f \in C([0, L])$.

Problem 2: Assume that $\mathcal{M}$ is a subset of $C([0,1])$ such that there are constants $m, L > 0$ such that $|f(\frac{1}{2})| \leq m$ for all $f \in \mathcal{M}$ and

$$|f(x) - f(\tilde{x})| \leq L|x - \tilde{x}| \text{ for all } f \in \mathcal{M} \text{ and all } x, \tilde{x} \in [0,1].$$

Show that $\mathcal{M}$ is relatively compact in $C([0,1])$.

Problem 3: Let $(E, \|\cdot\|)$ be a normed space. Set

$$T(x) = \begin{cases} x & \text{if } \|x\| \leq 1, \\ \frac{1}{\|x\|}\, x & \text{if } \|x\| > 1. \end{cases}$$

1. Show that $\|T(x) - T(\tilde{x})\| \leq 2\|x - \tilde{x}\|$ for all $x, \tilde{x} \in E$.
2. Show that the constant 2 cannot be improved in general. Take $E = \mathbb{R}^2$ with $\|\cdot\| = \|\cdot\|_{l^1}$.
3. What can be said if E is a Hilbert space with the norm given by the inner product?

Problem 4: Let E denote the real vector space $C([0,1])$ equipped with the norm $\|f\| = \max_{x\in[0,1]} |f(x)|$ and let $T : E \to E$ be a linear mapping. Assume that $Tf(x) \geq 0$ for all $x \in [0,1]$ provided $f(x) \geq 0$ for all $x \in [0,1]$. Show that

1. T is continuous
2. $\|T\| = \sup_{x\in[0,1]} T\mathbf{1}(x)$ where $\mathbf{1}$ denotes the constant function taking the value 1.
3. Let $U : E \to E$ be defined by $Uf(x) = \int_0^1 e^{xt} f(t)\, dt$ and the sequence $U_n : E \to E,\ n = 1, 2, 3, \ldots$ be defined by

   $$U_n f(x) = \int_0^1 \Sigma_{k=0}^n \frac{(tx)^k}{k!} f(t)\, dt.$$

   Prove that $\|U_n - U\| \to 0$.