TMA401/MMA400 Functional Analysis 2017/2018
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# Homework assignment 3

Deadline 2017-10-12

**Problem 1:** Set

$$A(f)(x) = \int_0^\pi \sin(x-y)\, f(y)\, dy, \quad 0 \le x \le \pi.$$

Find the operator norm of $A$ regarded as an operator on

1. the Banach space $C([0,\pi])$,
2. the Hilbert space $L^2([0,\pi])$.

**Problem 2:** Assume that $T$ is a self-adjoint operator on a Hilbert space $E$ that satisfies $T^3 = T^2$. Show that $T$ is an orthogonal projection operator. Is that true if $T$ instead satisfies $T^4 = T^3$?

**Problem 3:** Let $E$ be a complex Hilbert space with a complete ON-sequence $(e_n)_{n=1}^\infty$. Let $(\alpha_n)_{n=1}^\infty$ be a sequence of complex numbers and set

$$A(x) = \Sigma_{n=1}^\infty \alpha_n \langle x, e_n \rangle e_n, \ x \in E.$$

Give necessary and sufficient conditions on the sequence $(\alpha_n)_{n=1}^\infty$ such that

1. $A(x) \in E$ for every $x \in E$,
2. $A \in \mathcal{B}(E,E)$ and give an expression for $\|A\|$ in terms of the $\alpha_n$:s,
3. $A$ self-adjoint,
4. $A \in \mathcal{K}(E,E)$ (i.e. $A$ compact),
5. $A$ 1-1.

All statements in the five cases must be proved.

**Problem 4:** Analyse existence and uniqueness for solutions to the following BVP:

$$\begin{cases} u''(x) - u(x) = \lambda \arctan(u(x^2)), & 0 \le x \le 1 \\ u(0) = 2, u(1) = 1, & u \in C^2([0,1]) \end{cases}$$

Here $\lambda$ is a real number and all functions are real-valued. What can be said for different values of $\lambda$? Note that the boundary conditions are not homogeneous.