Homework assignment 3

① $A(f)(x) = \int_0^{\pi} \sin(x-y) f(y)\,dy$, $x \in [0,\pi]$

Calculate $\|A\|_{C([0,\pi]) \to C([0,\pi])}$ and $\|A\|_{L^2([0,\pi]) \to L^2([0,\pi])}$

1) $E = C([0,\pi])$ with $\|f\| = \max_{x\in[0,\pi]} |f(x)|$.

From $|A(f)(x)| \le \int_0^{\pi} |\sin(x-y)| \cdot |f(y)|\,dy \le \int_0^{\pi} |\sin(x-y)|\,dy \cdot \|f\| =$

$= 2\|f\|$ we conclude $\|A\| \le 2$.

For $f=1$ we note $A(f)(\pi) = \int_0^{\pi} \sin(\pi - y)\,dy = 2$

and hence $\|A\| \ge 2$. <u>Answer</u>: $\|A\| = 2$

2) $E = L^2([0,\pi])$ with $\|f\|_{L^2} = \left(\int_0^{\pi} |f(x)|^2 dx\right)^{1/2}$.

We observe that iA is a self-adjoint compact operator on the Hilbert space $L^2([0,\pi])$. Hence

$\|A\| = \|iA\| = \sup\{|\lambda| \in [0,\infty) : \lambda \text{ eigenvalue of } iA\}$.

Since $\sin(x-y) = \sin x \cdot \cos y - \cos x \cdot \sin y$ every eigenfunction must have the form $a\cos x + b\sin x$

Inserting this into $A(f) = \lambda f$, $f \ne 0$ gives $\lambda = \pm \frac{\pi}{2}$

<u>Answer</u>: $\|A\| = \frac{\pi}{2}$

② T self-adjoint operator on a Hilbert space E (clearly $T \in \mathcal{B}(E,E)$). Show that T is an orthogonal projection operator if $T^3 = T^2$. Show that T is an orthogonal projection operator if $T^4 = T^2$ instead.

Solution: $T \in \mathcal{B}(E,E)$ is an orthogonal projection operator iff T is self-adjoint and $T^2 = T$.

Assume $T^3 = T^2$: For $x \in E$ we have

$\|(T^2 - T)(x)\|^2 = \langle (T^2-T)(x), (T^2-T)(x)\rangle = \{T \text{ self-adjoint}\} =$

$= \langle \underbrace{(T^3 - T^2)(x)}_{=0}, (T-I)(x)\rangle = 0$

Hence $(T^2-T)(x) = 0$ for all $x \in E$ and so $T^2 = T$.

Assume $T^4 - T^3 = 0$: For $x \in E$ we have

$\|(T^3 - T^2)(x)\|^2 = \langle (T^3 - T^2)(x), (T^3 - T^2)(x)\rangle =$

$= \langle \underbrace{(T^4 - T^3)(x)}_{=0}, (T^2 - T)(x) \rangle = 0$

Hence $(T^3 - T^2)(x) = 0$ for all $x \in E$ and so $T^3 = T^2$.

The previous result gives $T^2 = T$. ////

(3) $(e_n)_{n=1}^{\infty}$ complete ON-sequence for Hilbert space E.

$(\alpha_n)_{n=1}^{\infty}$ sequence of complex numbers.

Set $A(x) = \sum_{n=1}^{\infty} \alpha_n \langle x, e_n \rangle e_n$, $x \in E$

1) $A(x) \in E$ for every $x \in E \iff \sup_n |\alpha_n| < \infty$

Pf: $\Leftarrow$: Assume $\sup_n |\alpha_n| \equiv M < \infty$. Fix $x \in E$.

Then $\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2$ by Parseval's formula.

Moreover $(\alpha_n \langle x, e_n \rangle)_{n=1}^{\infty} \in l^2$ since

$\sum_{n=1}^{\infty} |\alpha_n \langle x, e_n \rangle|^2 \leq M^2 \sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2 = M^2 \|x\|^2 < \infty$

Then $A(x) \in E$.

$\Rightarrow$: Assume $\sup_n |\alpha_n| = \infty$. Then there exists

a sequence $n_1 < n_2 < \ldots < n_k < \ldots$ s.th

$|\alpha_{n_k}| \geq k$, $k = 1, 2, \ldots$ Set $x = \sum_{k=1}^{\infty} \frac{1}{k} e_{n_k}$.

Then $x \in E$ since $\sum_{k=1}^{\infty} (\frac{1}{k})^2 < \infty$ but

$\sum_{k=1}^{\infty} \alpha_{n_k} \frac{1}{k} e_{n_k} \notin E$ since $\sum_{k=1}^{\infty} |\alpha_{n_k} \cdot \frac{1}{k}|^2 = \infty$

Hence $A(x) \notin E$.

2) $A \in \mathcal{B}(E, E) \iff \sup_n |\alpha_n| < \infty$. Moreover if $A \in \mathcal{B}(E, E)$ then $\|A\| = \sup_n |\alpha_n|$.

Pf: The equivalence is straight-forward and omitted.

Assume $A \in \mathcal{B}(E, E)$. Clearly $\|A\| \leq \sup_n |\alpha_n|$.

Moreover $\|A(e_n)\| = |\alpha_n| \|e_n\|$ $n = 1, 2, \ldots$ and so

$\|A\| \geq \sup_n |\alpha_n|$. This yields $\|A\| = \sup_n |\alpha_n|$.

3) A self-adjoint $\iff \alpha_n \in \mathbb{R}$ all n

Pf: A is assumed to be in $\mathcal{B}(E, E)$.

Fix $x, y \in E$. $\langle A(x), y \rangle = \langle \sum_{n=1}^{\infty} \alpha_n \langle x, e_n \rangle e_n, \sum_{k=1}^{\infty} \langle y, e_k \rangle e_k \rangle =$

$= \dots = \lim_{n\to\infty} \sum_{k=1}^{n} \langle \alpha_k \langle x, e_k\rangle e_k, \langle y, e_k\rangle e_k \rangle =$

$= \lim_{n\to\infty} \sum_{k=1}^{n} \langle \langle x, e_k\rangle e_k, \overline{\alpha}_k \langle y, e_k\rangle e_k \rangle = \dots =$

$= \langle \sum_{n=1}^{\infty} \langle x, e_n\rangle e_n, \sum_{k=1}^{\infty} \overline{\alpha}_k \langle y, e_k\rangle e_k \rangle$

Hence $A^*(x) = \sum_{n=1}^{\infty} \overline{\alpha}_n \langle x, e_n\rangle e_n$

So $(A-A^*)(x) = 0 \iff \sum_{n=1}^{\infty} (\alpha_n - \overline{\alpha}_n)\langle x, e_n\rangle e_n = 0$

which gives $A = A^* \iff \alpha_n = \overline{\alpha}_n$ all $n \iff \alpha_n \in \mathbb{R}$ all n

4) $A \in \mathcal{K}(E,E) \iff \lim_{n\to\infty} \alpha_n = 0$

Pf: $\Rightarrow$: Assume A compact operator.

Since $e_n \rightharpoonup 0$ in E, since $(e_n)_{n=1}^{\infty}$ is an ON-seq.,

we have $A(e_n) \to A(0) = 0$ in E.

But $A(e_n) = \alpha_n e_n$ all n so

$|\alpha_n| = \|A(e_n)\| \to 0$, $n \to \infty$

$\Leftarrow$: Assume $\alpha_n \to 0$, $n \to \infty$. Define $A_n : E \to E$

by $A_n(x) = \sum_{k=1}^{n} \alpha_k \langle x, e_k\rangle e_k$, $x \in E$.

Then $(A - A_n)(x) = \sum_{k=n+1}^{\infty} \alpha_k \langle x, e_k\rangle e_k$, $x \in E$ and

by 2) above $\|A - A_n\| = \sup_{k>n} |\alpha_k|$ $n = 1, 2, \dots$

and hence $\|A - A_n\| \to 0$, $n \to \infty$.

But A_n has $\dim \mathcal{R}(A_n) \le n$, i.e. A_n is finite rank operator, for all n so $A_n \in \mathcal{K}(E,E)$. Since $\mathcal{K}(E,E)$ is a <u>closed</u> subspace of $\mathcal{B}(E,E)$ we conclude that $A \in \mathcal{K}(E,E)$.

5) A 1-1 $\iff \alpha_n \neq 0$ all n.

Pf: Assume $A \in \mathcal{B}(E,E)$ and $A(x_1) = A(x_2)$.

Then $\sum_{n=1}^{\infty} \alpha_n (\langle x_1, e_n\rangle - \langle x_2, e_n\rangle) e_n = 0$ which is equivalent to $\sum_{n=1}^{\infty} \alpha_n \langle x_1 - x_2, e_n\rangle e_n = 0$ which is equivalent to $\alpha_n \langle x_1 - x_2, e_n\rangle = 0$ for all n.

This implies that $\forall x_1, x_2 \in E : A(x_1) = A(x_2) \Rightarrow x_1 = x_2$ is equivalent to $\alpha_n \neq 0$ all n.

④ $$\begin{cases} u''(x) - u(x) = \lambda \arctan(u(x^2)), & x \in [0,1] \\ u(0) = 2, \quad u(1) = 1. \end{cases} \qquad (+)$$

Show that the BVP has a solution for all $\lambda \in \mathbb{R}$ and that this solution is unique if $|\lambda|$ small.

Solution: <u>Step 1:</u> Remove the inhomogeneity in BC.

Set e.g. $v(x) = 2 - x$, which satisfies $v(0) = 2$ and $v(1) = 1$.

Set $u(x) = w(x) + v(x)$. Then (+) can be written

$$\begin{cases} w''(x) - w(x) = v(x) + \lambda \arctan(w(x^2) + v(x^2)) \\ w(0) = w(1) = 0 \end{cases} \qquad (*)$$

<u>Step 2:</u> Set $Lw = w'' - w$ with $R_1 w = w(0)$, $R_2 w = w(1)$.

Calculate the corresponding Green's function

A basis for $N(L)$ is given by e.g. $u_1(x) = e^x$, $u_2(x) = e^{-x}$

Set $g(x,t) = \theta(x-t)\, e(x,t) + b_1(t) u_1(x) + b_2(t) u_2(x)$,

where $e(x,t) = a_1(t) u_1(x) + a_2(t) u_2(x)$, satisfying

$$\begin{cases} e(t,t) = 0 : & a_1(t) e^{t} + a_2(t) e^{-t} = 0 & t \in [0,1] \\ e'_x(t,t) = 0 : & a_1(t) e^{t} - a_2(t) e^{-t} = 1 & \\ R_1 g(\cdot, t) = 0 : & b_1(t) + b_2(t) = 0 & t \in (0,1) \\ R_2 g(\cdot, t) = 0 : & e(1,t) + b_1(t) e + b_2(t) e^{-1} = 0 & \end{cases}$$

This implies

$$g(x,t) = \theta(x-t)\left(\tfrac{1}{2}\left(e^{x-t} - e^{t-x}\right)\right) + $$
$$+ \frac{1}{2} \frac{e}{e^2 - 1}\left(e^{x+t-1} - e^{x-t+1} + e^{-x-t+1} - e^{-x+t-1}\right)$$

Here one can observe that $g(x,t) \leq 0$ all $x, t \in [0,1]$.

<u>Step 3:</u> We can rewrite (*) as

$$w(x) = \int_0^1 g(x,t) \left[\lambda \arctan(w(x^2) + 2 - t^2) + 2 - t^2\right] dt$$

Set the RHS $= T(w)(x)$. Then

$T: C([0,1]) \to C([0,1])$, where $C([0,1])$ is equipped with the max-norm so that $C([0,1])$ becomes a Banach space.

If T is a contraction on $C([0,1])$ we get that T has a unique fixed point $w \in C([0,1])$ i.e. $w(x) = T(w)(x)$ where the RHS $\in C^2([0,1])$ and satisfies the homogeneous BC.

Fix $w_1, w_2 \in C([0,1])$.

$|T(w_1)(x) - T(w_2)(x)| \leq |\lambda| \int_0^1 |g(x,t)| dt \cdot \|w_1 - w_2\|$

where we used the mean value theorem to get

$|\arctan(w_1(t^2) + 2 - t^2) - \arctan(w_2(t^2) + 2 - t^2)| \leq |w_1(t^2) - w_2(t^2)| \leq$

$\leq \|w_1 - w_2\|$.

We see that T is a contraction on $C([0,1])$ if

$|\lambda| \int_0^1 |g(x,t)| dt \leq c < 1$ for all $x \in [0,1]$

for some $c<1$. To get an estimate for $|\lambda|$ we note that $g(x,t) \leq 0$ all $x,t \in [0,1]$. We have

$\int_0^1 |g(x,t)| dt = \int_0^1 g(x,t) \cdot (-1) dt \equiv j(x)$

where $j'' - j = -1$, $j(0) = j(1) = 0$ i.e. $j(x) = -1 - \frac{e^x + e^{1-x}}{e+1}$

where $\max_{x \in [0,1]} j(x) = j(\frac{1}{2}) = \frac{(\sqrt{e}-1)^2}{e+1}$

Conclusion: Banach's fixed point theorem implies that T has a unique fixed point for $|\lambda| < \frac{e+1}{(\sqrt{e}-1)^2}$, and hence $(+)$ has a unique solution $u \in C^2([0,1])$.

For $|\lambda| \geq \frac{e+1}{(\sqrt{e}-1)^2}$ we use Schauder's fixed point theorem to get the existence of a solution $u \in C^2([0,1])$ for $(+)$ (without uniqueness claim).

For $w \in C([0,1])$ we have $\|T(w)\| \leq \frac{(\sqrt{e}-1)^2}{e+1} \cdot (|\lambda| \frac{\pi}{2} + 2) \equiv N$

Set $S = \{w \in C([0,1]) : \|w\| \leq N\}$ which is a closed convex set in $C([0,1])$. $T(S) \subset S$ so $T(S)$ is (uniformly) bounded. Moreover, $T: S \to S$ is continuous, since

$\|T(w_1) - T(w_2)\| \leq |\lambda| \frac{(\sqrt{e}-1)^2}{e+1} \|w_1 - w_2\|$ for $w_1, w_2 \in S$,

and equicontinuity for the the functions in $T(S)$

follows from the fact that $g \in C([0,1] \times [0,1])$, and hence uniformly continuous on $[0,1]\times[0,1]$, and

$$|T(w)(x_1) - T(w)(x_2)| \leq N \int_0^1 |g(x_1,t) - g(x_2,t)|\,dt \quad , \; x_1, x_2 \in [0,1]$$

and $w \in C([0,1])$. We can then conclude that $T(S)$ is relatively compact in $C([0,1])$ from Arzela-Ascoli thm.
Schauder's (generalised) fixed point theorem implies that $T: S \to S$ has a fixed point, and hence (+) has a solution $u \in C^2([0,1])$. (not necessarily unique).