MATEMATIK
Chalmers tekniska högskola och Göteborgs universitet

Functional Analysis ENM, TMA401/ Applied Functional Analysis GU, MMA400,
Date: 2011-10-19 (4 hours)

Aids: Just pen, ruler and eraser.
Teacher on duty: Adam Andersson, 0703-088304

Note:	Write your name and personal number on the cover.
	Write your code on every sheet you hand in.
	Only write on one page of each sheet. Do not use red pen.
	Do not answer more than one question per page.
	State your methodology carefully. Write legibly.
	Questions are not numbered by difficulty.
	Sort your solutions by the order of the questions.
	Mark on the cover the questions you have answered.
	Count the number of sheets you hand in and fill in the number on the cover.
	To pass requires 10 points.

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1. Show that the following boundary value problem

$$u''(x) + u'(x) + 2\arctan(u^2(\sqrt{x})) = 0, \quad 0 \le x \le 1,$$
$$u(0) = u(1) = 1,$$
$$u \in C^2([0,1])$$

   has a unique solution.

   (4p)

2. Set

$$Tf(x) = \int_0^1 \sinh(x - t)f(t)\,dt, \ \ 0 \le x \le 1.$$

   Show that T is a linear bounded and compact operator when T is considered as an operator on the Banach spaces

   (a) $C([0,1])$
   (b) $L^2([0,1])$

   respectively (with the standard norms). Also calculate the operator norms.

   (4p)

P.T.O

3. Let $T : \{x \in \mathbb{R}^n : \|x\| \leq 1\} \rightarrow \mathbb{R}^n$ be a continuous mapping. Moreover assume that $\langle T(x), x \rangle > 0$ for all x with $\|x\| = 1$. Here $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^n$ with the induced norm $\| \cdot \|$. Prove[1] that there exists a $x_0 \in \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ such that $T(x_0) = 0$.

(4p)

4. Formulate the *Method of continuity*. Prove the statement.

(5p)

5. Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and $A \in \mathcal{B}(H, H)$. Define the adjoint operator A^*, show that it is a uniquely defined mapping in $\mathcal{B}(H, H)$ and that $\|A^*\| = \|A\|$.

(4p)

6. Show that it is impossible to equip $C([0,1])$ with an inner product in such a way that the norm induced by the inner product is equal to the standard norm $\|f\| = \max_{x \in [0,1]} |f(x)|$ for $f \in C([0,1])$.

(4p)

For information on the announcement of results see the course homepage where also solutions to the problems will be presented.

GOOD LUCK!　　　　　　　PK

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[1]Hint: Consider the mapping $G(x) = x - \epsilon T(x)$ for some properly choosen $\epsilon > 0$.