**MATEMATIK**
**Chalmers tekniska högskola och Göteborgs universitet**

Functional Analysis ENM, TMA401/ Applied Functional Analysis GU, MMA400,
Date: 2012-01-12 (4 hours)

Aids: Just pen, ruler and eraser.
Teacher on duty: Emil Gustavsson, 0703-088304

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| **Note:** | Write your name and personal number on the cover.<br>Write your code on every sheet you hand in.<br>Only write on one page of each sheet. Do not use red pen.<br>Do not answer more than one question per page.<br>State your methodology carefully. Write legibly.<br>Questions are not numbered by difficulty.<br>Sort your solutions by the order of the questions.<br>Mark on the cover the questions you have answered.<br>Count the number of sheets you hand in and fill in the number on the cover.<br>To pass requires 10 points. |

1. Show that the following boundary value problem

$$\begin{aligned} &u''(x) - u(x) + \tfrac{1}{2}(1 + u(x^2)) = 0, \quad 0 \le x \le 1, \\ &u(0) = u'(0) = 0, \\ &u \in C^2([0,1]) \end{aligned}$$

has a unique solution.

(4p)

2. For $f \in L^2([0,1])$ set

$$Tf(x) = \int_{\sqrt{x}}^{1} \frac{1}{x+t} f(t)\, dt, \quad 0 \le x \le 1.$$

Show that $T$ is a bounded linear operator on $L^2([0,1])$ and calculate $T^*$. Is $T$ a compact operator?

(4p)

3. Let $(e_n)_{n=1}^{\infty}$ be an ON-basis in a complex Hilbert space $H$ with norm $\|\cdot\|$. Assume that $(f_n)_{n=1}^{\infty}$ is a sequence in $H$ with the properties

   (a) $\sup_{n=1,2,3,\dots} \|f_n\| < \infty$

   (b) $f_n \in \{e_1, e_2, \dots, e_n\}^{\perp}$ for all $n$.

Show that $(f_n)_{n=1}^{\infty}$ converges weakly in $H$.

(4p)

4. Formulate and prove the Riesz representation theorem.

(5p)

5. State and prove[1] the orthogonal projection theorem.

(4p)

6. Prove the following statements:

   (a) For $p, r \in [1, \infty)$ and $f \in C([0,1] \times [0,1])$ define

   $$\|f\|_{L^{r,p}} = (\int_0^1 (\int_0^1 |f(x,y)|^p \, dy)^{\frac{r}{p}} \, dx)^{\frac{1}{r}}.$$

   Then $\| \cdot \|_{L^{r,p}}$ defines a norm on $C([0,1] \times [0,1])$.

   (b) For $p \in (0, \infty)$ and $f \in C([0,1])$ define

   $$\|f\|_{L^p} = (\int_0^1 |f(x)|^p \, dx)^{\frac{1}{p}}.$$

   If $\| \cdot \|_{L^p}$ defines a norm on $C([0,1])$ then[2] $p \geq 1$.

(4p)

For information on the announcement of results see the course homepage where also solutions to the problems will be presented.

GOOD LUCK! PK

[1]If the "closest point property-proposition" is used it should be stated and proved.
[2]Hint: The inequality $(a+b)^p < a^p + b^p$ for $a, b > 0$ and $0 < p < 1$ might be useful.