MATEMATIK
Chalmers tekniska högskola och Göteborgs universitet

Functional Analysis ENM, TMA401/ Applied Functional Analysis GU, MMA400,
Date: 2014-08–30 (4 hours)

Aids: Just pen, ruler and eraser.
Teacher on duty: Åse Fahlander, 0703-088304

Note:	Write your name and personal number on the cover. Write your code on every sheet you hand in. Only write on one page of each sheet. Do not use red pen. Do not answer more than one question per page. State your methodology carefully. Write legibly. Questions are not numbered by difficulty. Sort your solutions by the order of the questions. Mark on the cover the questions you have answered. Count the number of sheets you hand in and fill in the number on the cover. To pass requires 10 points.

1. Show that the following boundary value problem

$$\begin{cases} u''(x) + u(x) = \dfrac{u(x)}{2+u^2(x)}, \quad x \in [0, \dfrac{\pi}{2}], \\ \\ u(0) = u(\frac{\pi}{2}) = 0, \; u \in C^2([0, \frac{\pi}{2}]). \end{cases}$$

has a unique solution u.

(4p)

2. Let A be a positive compact self-adjoint operator on a Hilbert space H with operator norm ≤ 1. Give an upper estimate for the operator norm of $3A^4 - 20A^3 + A^2$ (better than the trivial estimate 24).

(3p)

3. Let k be a non-zero continuous function on $[-\pi, \pi]$ and define the operator $T \in \mathcal{B}(L^2([-\pi, \pi]))$ by $Tf(x) = k(x)f(x)$. Show that T is not compact.

(4p)

P.T.O.

4. State and prove Banach´s fixed point theorem.

(5p)

5. Let $k(x,y) \in C([0,1] \times [0,1])$ and define

$$Af(x) = \int_0^1 k(x,y) f(y)\, dy, \quad x \in [0,1].$$

Show that A is a compact operator on $L^2([0,1])$ and also on $C([0,1])$.

(5p)

6. Let $C_n, n = 1, 2, 3, \ldots$, be a sequence of closed convex subsets in a Hilbert space H. Moreover assume that

$$C_1 \supset C_2 \supset \ldots \supset C_n \supset \ldots$$

and that

$$C = \bigcap_{n=1}^{\infty} C_n \neq \emptyset.$$

For $x \notin C$, let $x_n \in C_n$ be defined by

$$\|x - x_n\| = \inf_{y \in C_n} \|x - y\|$$

for $n = 1, 2, 3, \ldots$. Show that $x_n \to \tilde{x}$ in H and give a geometric interpretation of $\tilde{x}$.

(4p)

For information on the announcement of results see the course homepage where also solutions to the problems will be presented.

GOOD LUCK! PK