MATEMATIK
Chalmers tekniska högskola och Göteborgs universitet

Functional Analysis ENM, TMA401/ Applied Functional Analysis GU, MMA400,
Date: 2014-10–29 (4 hours)

Aids: Just pen, ruler and eraser.
Teacher on duty: Peter Kumlin, 7723532

Note:	Write your name and personal number on the cover. Write your code on every sheet you hand in. Only write on one page of each sheet. Do not use red pen. Do not answer more than one question per page. State your methodology carefully. Write legibly. Questions are not numbered by difficulty. Sort your solutions by the order of the questions. Mark on the cover the questions you have answered. Count the number of sheets you hand in and fill in the number on the cover. To pass requires 10 points.

1. Consider the differential operator $L = (\frac{d}{dx})^2$ defined on $C^2([0,1])$ with boundary conditions $u(0) = u'(1)$ and $u'(0) = u(1)$. Calculate
   (a) the corresponding Green´s function $g(x,t)$, and
   (b) prove that the boundary value problem
   $$\begin{cases} u''(x) = \sin(\sqrt{|u(x^2)| + 1}), & x \in [0,1], \\ u(0) = u'(1), u'(0) = u(1) \end{cases}$$
   has a unique solution $u \in C^2([0,1])$.

(4p)

2. For $f \in L^2([0,1])$ and $x \in [0,1]$ set
   $$A(f)(x) = \int_0^1 (x-y) f(y)\, dy.$$
   Show that
   (a) $A(f) \in L^2([0,1])$ for $f \in L^2([0,1])$
   (b) A is a bounded linear operator on $L^2([0,1])$, and
   (c) calculate $\|A\|$ and also
   (d) $\|(A - \frac{1}{2\sqrt{3}} I)^{10}\|$ where I denotes the identity operator on L^2.

(5p)

3. Let $g \in L^2([0,1])$ be a fixed function and consider the equation

$$(x+1)\int_0^1 tf(t)\,dt = f(x) + g(x), \quad x \in [0,1]$$

for $f \in L^2([0,1])$. Show that this equation has a unique solution.

(3p)

4. Let X be a Banach space and assume that $A \in \mathcal{B}(X,X)$ with operator-norm $\|A\| < 1$. Show that $(I+A)^{-1}$ exists as a mapping $X \to X$ and belongs to $\mathcal{B}(X,X)$.
Moreover define $\sigma(A)$, the spectrum of A, and $\rho(A)$, the resolvent set of A, and show that $\sigma(A)$ is a compact set in $\mathbb{C}$.

(5p)

5. Assume that $x_n \rightharpoonup x$ in a Hilbert space $(H, \langle\cdot,\cdot\rangle)$. Show that
   (a) $A \in \mathcal{B}(H)$ implies that $A(x_n) \rightharpoonup A(x)$ in H, and that
   (b) $A \in \mathcal{K}(H)$ implies that $A(x_n) \to A(x)$ in H.

(4p)

6. Let X be a Banach space and A a non-empty subset of a normed space Y. Let

$$T : X \times A \to X$$

be a continuous mapping[1] and assume that there exits a $k \in [0,1)$ such that

$$\|T(x_1,y) - T(x_2,y)\| \leq k\|x_1 - x_2\| \text{ for all } x_1, x_2 \in X \text{ and } y \in A.$$

Show that for each fixed $y \in A$ the mapping T has a unique fixed point $x_0(y)$ and that

$$x_0 : A \to X$$

is a continuous mapping.

(4p)

For information on the announcement of results see the course homepage where also solutions to the problems will be presented.

GOOD LUCK! PK

[1]This means that $x_n \to x$ in X and $y_n \to y$ in Y implies that $T(x_n, y_n) \to T(x, y)$ in X.