MATEMATIK
Chalmers tekniska högskola och Göteborgs universitet

Functional Analysis ENM, TMA401/ Applied Functional Analysis GU, MMA400,
Date: 2015-01–05 (4 hours)

Aids: Just pen, ruler and eraser.
Teacher on duty: Tim Cardilin, 0703-088304

Note:	Write your name and personal number on the cover. Write your code on every sheet you hand in. Only write on one page of each sheet. Do not use red pen. Do not answer more than one question per page. State your methodology carefully. Write legibly. Questions are not numbered by difficulty. Sort your solutions by the order of the questions. Mark on the cover the questions you have answered. Count the number of sheets you hand in and fill in the number on the cover. To pass requires 10 points.

1. Consider the differential operator $L = (\frac{d}{dx})^2$ defined on $C^2([0,1])$ with boundary conditions $u(0) = u'(1)$ and $u'(0) = u(1)$. Calculate
   (a) the corresponding Green´s function $g(x,t)$, and
   (b) prove that the boundary value problem
   $$\begin{cases} u''(x) = \sin(\sqrt{|u(x^2)| + 1}), \quad x \in [0,1], \\ u(0) = u'(1), u'(0) = u(1) \end{cases}$$
   has a unique solution $u \in C^2([0,1])$.

   (4p)

2. Let
   $$M = \{f \in L^2([0,1]) : \int_0^1 f(x)\,dx = \int_0^1 xf(x)\,dx = \int_0^1 x^2 f(x)\,dx = 0\}.$$
   Given $h \in L^2([0,1])$ find a formula for the function in M which is closest to h (in the $L^2([0,1])$-norm).

   (4p)

3. Assume that $k : [0,\infty) \to \mathbb{R}$ is a continuous function with
   $$\int_0^\infty |k(y)|\,dy < \infty.$$
   Set
   $$Kf(x) = \int_0^\infty k(x+y)f(y)\,dy, \quad x \in [0,\infty).$$
   Show that K is a bounded linear operator on $L^2([0,\infty))$. Also show that K is compact.

(4p)

4. State and prove Banach´s fixed point theorem.

(5p)

5. Let T be a bounded linear operator on a Hilbert space H. Show that
$$\overline{\mathcal{R}(T^*)} = \mathcal{N}(T)^\perp.$$

(4p)

6. Let H be a Hilbert space and T a compact self-adjoint operator on H with a complete ON-sequence $(e_n)_{n=1}^\infty$ of eigenvectors corresponding to the eigenvalues $(\lambda_n)_{n=1}^\infty$. Show that if $\lambda \neq 0$ and $\lambda \neq \lambda_n$, $n = 1, 2, 3, \ldots$ then $\lambda I - T$ is invertible and
$$(\lambda I - T)^{-1}(x) = \frac{1}{\lambda}(x + \Sigma_{n=1}^\infty \frac{\lambda_n}{\lambda - \lambda_n} \langle x, e_n \rangle e_n), \quad x \in H.$$

(4p)

For information on the announcement of results see the course homepage where also solutions to the problems will be presented.

GOOD LUCK! PK