MATEMATIK
Chalmers tekniska högskola och Göteborgs universitet

Functional Analysis ENM, TMA401/ Applied Functional Analysis GU, MMA400,
Date: 2015-10–28 (4 hours)

Aids: Just pen, ruler and eraser.
Teacher on duty: Peter Kumlin, 7723532

Note:	Write your name and personal number on the cover. Write your code on every sheet you hand in. Only write on one page of each sheet. Do not use red pen. Do not answer more than one question per page. State your methodology carefully. Write legibly. Questions are not numbered by difficulty. Sort your solutions by the order of the questions. Mark on the cover the questions you have answered. Count the number of sheets you hand in and fill in the number on the cover. To pass requires 10 points.

1. Show that the BVP

$$\begin{cases} u''(x) + 2u'(x) + u(x) = \dfrac{1}{2}\sin^2(u(x)), & 0 \le x \le 1 \\ u(0) = u(1) = 0 \end{cases}$$

has a unique solution $u \in C^2([0,1])$.

(4p)

2. Let $a(x)$ be a continuous complex-valued function on $[0,1]$ and define $A : L^2([0,1]) \to L^2([0,1])$ by

$$A(f)(x) = a(x)f(x), \quad x \in [0,1].$$

Show that A is a bounded linear operator on $L^2([0,1])$ and calculate $\|A\|$.

(4p)

3. Assume that $(e_n)_{n=1}^{\infty}$ is a complete ON-sequence on a Hilbert space E and that $(\alpha_n)_{n=1}^{\infty}$ is a bounded sequence of complex numbers. Set

$$A(x) = \Sigma_{n=1}^{\infty}\alpha_n\langle x, e_n\rangle e_n, \quad x \in E.$$

You may assume without proof that A defines a bounded linear operator on E.

(a) Calculate the eigenvalues for A

(b) Give a necessary and sufficient conditions on $(\alpha_n)_{n=1}^{\infty}$ for A to be onto (=surjective) and prove your statement.

(c) Give an example of $(\alpha_n)_{n=1}^{\infty}$ with the property that $\mathcal{R}(A)$ is a dense proper subspace of E. Prove your statement.

(4p)

4. State and prove Hölder´s inequality for sequence spaces l^p.

(5p)

5. Let E be a Hilbert space and let $\mathcal{K}(E, E)$ denote the vector space of all compact linear operators on E. Show that $\mathcal{K}(E, E)$ is closed in $\mathcal{B}(E, E)$. Here $\mathcal{B}(E, E)$ denotes the vector space af all bounded linear operators on E equipped with the operator norm.

(4p)

6. Let X be a Banach space and let $T : X \to X$ be a contraction on X. Moreover assume that $(T_n)_{n=1}^{\infty}$ is a sequence of mappings $X \to X$ such that

$$\lim_{n\to\infty} \sup_{x\in X} \|T_n(x) - T(x)\| = 0.$$

Assume that for each n there exists at least one $x_n \in X$ such that $T_n(x_n) = x_n$. Show that the sequence $(x_n)_{n=1}^{\infty}$ converges. What can be said about the limit element?

(4p)

For information on the announcement of results see the course homepage where also solutions to the problems will be presented.

GOOD LUCK! PK