MATEMATIK
Chalmers tekniska högskola och Göteborgs universitet

Functional Analysis ENM, TMA401/ Applied Functional Analysis GU, MMA400,
Date: 2017–01–05 (4 hours)

Aids: Just pen, ruler and eraser.
Teacher on duty: Peter Kumlin

Note:	Write your name and personal number on the cover. Write your code on every sheet you hand in. Only write on one page of each sheet. Do not use red pen. Do not answer more than one question per page. State your methodology carefully. Write legibly. Questions are not numbered by difficulty. Sort your solutions by the order of the questions. Mark on the cover the questions you have answered. Count the number of sheets you hand in and fill in the number on the cover. To pass requires 10 points.

1. Show that the BVP

$$\begin{cases} u''(x) + u(x) + \lambda \cos(1 + u(x)) = 0, \ x \in [0,1] \\ \\ u(0) = u'(0) = 0 \end{cases}$$

has a unique solution $u \in C^2([0,1])$ for $|\lambda| < \epsilon$, ϵ small. Give an upper bound on ϵ.

(4p)

2. Let X be a Banach space and $A : X \to X$ a bounded linear operator with $\|A^n\| < 1$ for some positive integer n. Show that $I - A$ is a bijection and that its inverse $(I - A)^{-1} : X \to X$ is continuous.

(4p)

3. Let E be a Hilbert space and $A : E \to E$ a bounded linear and self-adjoint operator that satisfies $A^3 = A^2$. Show that A is an orthogonal projection operator.

(4p)

4. Let E be a Hilbert space and $(x_n)_{n=1}^{\infty}$ a weakly converging sequence in E. Show that

$$\sup_{n=1,2,3,\ldots} \|x_n\| < \infty.$$

Give an example where $(x_n)_{n=1}^{\infty}$ converges weakly to x but

$$\|x_n\| \not\to \|x\| \text{ as } n \to \infty.$$

What can be said if $(x_n)_{n=1}^{\infty}$ converges weakly to x and

$$\|x_n\| \to \|x\| \text{ as } n \to \infty?$$

(5p)

5. Let $\mathcal{P}_n$ denote the vector space of all polynomials of degree at most n on $\mathbb{R}$, where n is a positive integer. Set

$$\|p\| = \Sigma_{k=0}^{n} |p(k)|, \quad p \in \mathcal{P}_n.$$

Show that $(\mathcal{P}_n, \|\cdot\|)$ is a Banach space.

(4p)

6. Let E be a separable Hilbert space and $A : E \to E$ a linear compact operator. Show that for every $\epsilon > 0$ there exists a finite-rank operator B such that $\|A - B\| < \epsilon$.

(4p)

For information on the announcement of results see the course homepage where also solutions to the problems will be presented.

GOOD LUCK! PK