MATEMATIK
Chalmers tekniska högskola och Göteborgs universitet

Functional Analysis ENM, TMA401/ Applied Functional Analysis GU, MMA400,
Date: 2018–01–04 (4 hours)

Aids: Just pen, ruler and eraser.
Teacher on duty: Andreas Petersson, 5325

Note: Write your name and personal number on the cover.
Write your code on every sheet you hand in.
Only write on one page of each sheet. Do not use red pen.
Do not answer more than one question per page.
State your methodology carefully. Write legibly.
Questions are not numbered by difficulty.
Sort your solutions by the order of the questions.
Mark on the cover the questions you have answered.
Count the number of sheets you hand in and fill in the number on the cover.
To pass requires 10 points.

---

1. Let $f \in C([0,1])$ and $\lambda \in \mathbb{R}$ with $|\lambda| < 8$. Show that the boundary value problem

$$\begin{cases} u'' - \lambda\sqrt{1+|u(x)|^2} = f(x), \ x \in [0,1] \\ \\ u(0) = u(1) = 0 \end{cases}$$

has a unique solution $u \in C^2([0,1])$.

(4p)

2. Let $T : C([0,1]) \to C([0,1])$ be defined by

$$Tf(x) = \int_0^x f(t)\,dt, \ x \in [0,1],$$

where $C([0,1])$ is equipped with the max-norm. Show that

(a) T is not a contraction,
(b) T has a unique fixed point,
(c) T^2 is a contraction, and
(d) what could be said about the convergence of the sequence of iterates $(T^n f)_{n=1}^{\infty}$ for a general $f \in C([0,1])$? Prove your statement.

(4p)

3. Let E be the Hilbert space l^2 with the standard inner product. For each $\mathbf{x} = (x_1, x_2, x_3, \ldots, x_n, \ldots) \in l^2$ consider the right and left shift operators

$$S_r\mathbf{x} = (0, x_1, x_2, x_3, \ldots, x_n, \ldots)$$

and

$$S_l x = (x_2, x_3, \dots, x_n, \dots)$$

respectively. Then

(a) determine the operator norms $\|S_r\|$ and $\|S_l\|$ and whether any of S_r, S_l is/are compact,

(b) show that S_r has no eigenvalues,

(c) show that $\sigma(S_r) = [-1, 1]$,

(d) show that every $\lambda \in (-1, 1)$ is an eigenvalue for S_l and determine the corresponding eigenspace and

(e) prove that $\sigma(S_l) = [-1, 1]$.

(4p)

4. State and prove Riesz representation theorem. Show that the assumption of "Hilbert space" in the theorem cannot be relaxed to "inner product space".

(5p)

5. Let X, Y be Banach spaces. We say that $T_n \to T$ uniformly if $\|T_n - T\| \to 0$ as $n \to \infty$ where $\|\cdot\|$ denotes the operator norm in $\mathcal{B}(X, Y)$. Moreover we say that $T_n \to T$ strongly if $T_n x \to Tx$ in Y for every $x \in X$. Show that

(a) uniform convergence in $\mathcal{B}(X, Y)$ implies strong convergence in $\mathcal{B}(X, Y)$, and

(b) give an example of a sequence $T_n \in \mathcal{B}(E, E)$, where E is a Hilbert space, that converges strongly but not uniformly in $B(E, E)$.

(4p)

6. Let $(x_n)_{n=1}^{\infty}$ be a weakly converging sequence in a Hilbert space E. Show that

$$\sup_{n=1,2,3,\dots} \|x_n\| < \infty.$$

Also show that if x denotes the weak limit af the sequence $(x_n)_{n=1}^{\infty}$ than

$$\|x\| \leq \lim_{n \to \infty} \inf_{k \geq n} \|x_k\|.$$

(4p)

For information on the announcement of results see the course homepage where also solutions to the problems will be presented.

GOOD LUCK! PK