Functional analysis TMA401/MMA400

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Course diary — What has happened?

Week 1 Discussion of introductory example, see section 1. Definition of real/complex vector space, remark on existence of unique zero vector and inverse vectors, example of real vector spaces (sequence spaces and function spaces). Hölder and Minkowski inequalities. Introducing (the to all students very well-known concepts) linear combination, linear independence, span of a set, (vector space-) basis (= Hamel basis) with examples. All vector spaces have basis (using Axiom of choice/Zorn's lemma; it was not proven but stated). Introducing norms on vector spaces with examples, equivalent norms, convergence of sequences in normed spaces, showed that $C([0,1])$ can be equipped with norms that are not equivalent. Stated and proved that all norms on finite-dimensional vector spaces are equivalent. A proof of this is supplied below. Also mentioned that all infinte-dimensional vector spaces can be equipped with norms that are not equivalent (easy to prove once we have a Hamel basis).

Theorem 0.1. *Suppose E is a vector space with $\dim(E) < \infty$. Then all norms on E are equivalent.*

PROOF: We observe that the relation that two norms are equivalent is transitive, so it is enough to show that an arbitrary norm $\|\cdot\|$ on E is equivalent to a fixed norm $\|\cdot\|_*$ on E. Let $x_1, x_2, \ldots, x_n$, where $n = \dim(E)$, be a basis for E. This means that for every $x \in E$ there are uniquely defined scalars $\alpha_k(x)$, $k = 1, 2, \ldots, n$, such that

$$x = \alpha_1(x)x_1 + \alpha_2(x)x_2 + \ldots + \alpha_n(x)x_n.$$

Set $\|x\|_* = |\alpha_1(x)| + |\alpha_2(x)| + \ldots |\alpha_n(x)|$ for $x \in E$.
Claim: $\|\cdot\|_*$ defines a norm on E
It is easy to prove that the three axioms for being a norm are satisfied (and is left as an exercise). We note that

$$\|x\| = \|\alpha_1(x)x_1 + \alpha_2(x)x_2 + \ldots + \alpha_n(x)x_n\| \leq \Sigma_{k=1}^n |\alpha_k(x)|\, \|x_k\| \leq \beta\|x\|_*,$$

where $\beta = \max_{k=1,2,\ldots,n} \|x_k\| > 0$. Hence we have proven that

$$\|x\| \leq \beta\|x\|_* \text{ for all } x \in E.$$

It remains to prove that there exists an $\alpha > 0$ such that

$$\alpha\|x\|_* \leq \|x\| \text{ for all } x \in E. \tag{1}$$

Let us argue by contradiction. Assume that (1) is false. Then there exists a sequence $(y_m)_{m=1}^\infty$ such that

$$\frac{1}{m}\|y_m\|_* > \|y_m\| \text{ for } k = 1, 2, \ldots$$

WLOG we can assume $\|y_m\|_* = 1$ for all m (note that in general $\alpha_k(\lambda x) = \lambda\alpha_k(x)$ for scalars λ, $x \in E$ and $k = 1, 2, \ldots, n$. Hence y_m can be replaced by λy_m for proper choice of λ). Now set

$$\mathbb{\alpha}_m = (\alpha_1^{(m)}, \alpha_2^{(m)}, \ldots, \alpha_n^{(m)}),\ m = 1, 2, 3, \ldots,$$

where

$$\alpha_k(y_m) = \alpha_k^{(m)}, \;\; k = 1, 2, \ldots, n, \; m = 1, 2, 3, \ldots$$

Since $|\alpha_1^{(m)}| \leq 1, \; m = 1, 2, 3, \ldots$ we conclude from Bolzano-Weierstrass theorem that there exists a converging subsequence, denoted $(\alpha_{1,1}^{(m)})_{m=1}^{\infty}$, of $(\alpha_1^{(m)})_{m=1}^{\infty}$. Call the limit element $\bar{\alpha}_1$. Now consider the sequence formed by the second coordinate in

$$\mathbb{a}_{m,1} = (\alpha_{1,1}^{(m)}, \alpha_{2,1}^{(m)}, \ldots, \alpha_{n,1}^{(m)}), \; m = 1, 2, 3, \ldots,$$

i.e. $(\alpha_{2,1}^{(m)})_{m=1}^{\infty}$. Since $|\alpha_{2,1}^{(m)}| \leq 1, \; m = 1, 2, 3, \ldots$ we conclude from Bolzano-Weierstrass theorem that there exists a converging subsequence, denoted $(\alpha_{2,2}^{(m)})_{m=1}^{\infty}$, of $(\alpha_{2,1}^{(m)})_{m=1}^{\infty}$. Call the limit element $\bar{\alpha}_2$. We proceed inductively. After n steps we have a sequence

$$\mathbb{a}_{m,n} = (\alpha_{1,n}^{(m)}, \alpha_{2,n}^{(m)}, \ldots, \alpha_{n,n}^{(m)}), \; m = 1, 2, 3, \ldots,$$

where we have that

$$\alpha_{l,n}^{(m)} \to \bar{\alpha}_l, \; m \to \infty$$

for $l = 1, 2, \ldots, n$. Set

$$z_m = \alpha_{1,n}^{(m)} x_1 + \alpha_{2,n}^{(m)} x_2 + \ldots + \alpha_{n,n}^{(m)} x_n, \; m = 1, 2, 3, \ldots$$

We have that each z_m is equal to a y_l for some $l \geq m$ and hence

$$\|z_m\| \to 0 \;\text{ as } m \to \infty.$$

Moreover set $\bar{z} = \bar{\alpha}_1 x_1 + \bar{\alpha}_2 x_2 + \ldots + \bar{\alpha}_n x_n$. We have

$$\|\bar{z}\| \leq \|\bar{z} - z_m\| + \|z_m\| =$$

$$= \|(\bar{\alpha}_1 - \alpha_{1,n}^{(m)}) x_1 + (\bar{\alpha}_2 - \alpha_{2,n}^{(m)}) x_2 + \ldots + (\bar{\alpha}_n - \alpha_{n,n}^{(m)}) x_n\| + \|z_m\| \leq$$

$$\leq |\bar{\alpha}_1 - \alpha_{1,n}^{(m)}| \, \|x_1\| + |\bar{\alpha}_2 - \alpha_{2,n}^{(m)}| \, \|x_2\| + \ldots + |\bar{\alpha}_n - \alpha_{n,n}^{(m)}| \, \|x_n\| + \|z_m\| \to 0$$

as $m \to \infty$. Hence $\|\bar{z}\| = 0$ and

$$\mathbb{0} = \bar{\alpha}_1 x_1 + \bar{\alpha}_2 x_2 + \ldots + \bar{\alpha}_n x_n \;\text{ with }\; |\bar{\alpha}_1| + |\bar{\alpha}_2| + \ldots + |\bar{\alpha}_n| = 1.$$

This contradicts the assumption that $x_1, x_2, \ldots, x_n$ is a basis for E since there must be an $\bar{\alpha}_k$ that is non-zero. We conclude that (1) above must hold. The theorem is proven.

$\square$

We see in the proof that one difficulty is to find a suitable notation for subsequences of subsequences of ... of sequences. In general, if we have a sequence a_n, $n = 1, 2, 3, \ldots$, a subsequence can be denoted by a_{n_k}, $k = 1, 2, 3, \ldots$ Here n_k, $k = 1, 2, 3, \ldots$ is an increasing sequence of positive integers. This mean that every element in the latter sequence appears in the first one and in the same order as there. If we want to denote a subsequence of a_{n_k}, $k = 1, 2, 3, \ldots$ we could write $a_{n_{k_l}}$, $l = 1, 2, 3, \ldots$ but that is not very efficient, in particular if we want to continue to extract subsequences. We can write $a_{n,1}$, $n = 1, 2, 3, \ldots$ instead of a_{n_k}, $k = 1, 2, 3, \ldots$ and correspondingly $a_{n,2}$, $n = 1, 2, 3, \ldots$ for $a_{n_{k_l}}$, $l = 1, 2, 3, \ldots$ and so on. The second subindex indicates which generation of taking subsequences we are considering.

Week 2 Definition and discussion of the topological concepts: open set/closed set and closure of a set, bounded/dense/compact sets. The theorem stating that all closed bounded sets in a normed space are compact is a characterisation of finite-dimensionality for normed spaces was stated and proved. Convergent sequences and Cauchy sequences were treated and Banach spaces were introduced. Example of Banach spaces and non-Banach spaces where given. It was proved in detail that $C([0,1])$ with max-norm and l^p, $1 \le p < \infty$ with the standard l^p-norm were Banach spaces. Also converging series and absolutely converging series in normed spaces were treated. Mappings between normed spaces – continuity, linearity, bounded linear mappings and how these properties for mappings are related was discussed. We introduced operator-norm for bounded linear mappings and showed that this was a norm for the vector space of all bounded linear mappings between two normed spaces E_1, E_2 which we denote $\mathcal{B}(E_1, E_2)$. We also proved that this normed space was a Banach space given that the target space E_2 was a Banach space. Examples/exercises on bounded linear mappings were discussed.

Finally we stated and proved the Banach-Steinhaus theorem (uniform boundedness principle). Since the proof differs from the one in DM we supply the proof presented at the lecture.

Theorem 0.2. *Assume that $(E_1, \|\cdot\|_1)$ is a Banach space and that $(E_2, \|\cdot\|_2)$ is a normed space. Moreover assume that*

$$\sup_{T\in\mathcal{F}} \|T(x)\|_2 < \infty \text{ for all } x \in E_1.$$

Then

$$\sup_{T\in\mathcal{F}} \|T\| < \infty.$$

PROOF: The proof is done in two steps.
Step 1: We prove the theorem under the assumption

$$\exists x_0 \in E_1\ \exists r > 0\ \exists M > 0\ \forall x \in \overline{B(x_0, r)}\ \forall T \in \mathcal{F} : \|T(x)\|_2 \le M. \tag{2}$$

Fix an arbitrary $T \in \mathcal{F}$. For $x \in E_1$ with $\|x\|_1 \le r$ we have

$$\|T(x)\|_2 = \|T(x_0 + x - x_0)\|_2 = \|T(x_0 + x) - T(x_0)\|_2 \le$$

$$\le \|T(x_0 + x)\|_2 + \|T(x_0)\|_2 \le 2M$$

where we used the linearity of T. Hence for $x \ne \mathbb{0}$ it follows that

$$2M \ge \|T(\frac{r}{\|x\|_1} x)\|_2 = \frac{r}{\|x\|_1}\|T(x)\|_2$$

where the linearity of T is used. Thus

$$\|T(x)\|_2 \le \frac{2M}{r}\|x\|_1 \text{ all } x \in E_1.$$

We conclude that

$$\sup_{T\in\mathcal{F}} \|T\| < \infty.$$

Step 2: Remains to prove that assumption (2) is true. We argue by contradiction. So assume that the negation of the statement (2) is true. Hence

$$\forall x_0 \in E_1 \; \forall r > 0 \; \forall M > 0 \; \exists x \in \overline{B(x_0, r)} \; \exists T \in \mathcal{F} : \|T(x)\|_2 > M.$$

This statement is equivalent to

$$\forall x_0 \in E_1 \; \forall r > 0 \; \forall M > 0 \; \exists x \in B(x_0, r) \; \exists T \in \mathcal{F} : \|T(x)\|_2 > M \tag{3}$$

since

$$B(x_0, r_1) \subset B(x_0, r_2) \subset \overline{B(x_0, r_2)} \subset B(x_0, r_3) \text{ for all } r_1 < r_2 < r_3.$$

The idea is to find a Cauchy sequence $(x_n)_{n=1}^{\infty}$ in E_1 (which converges since E_1 is a Banach space) and a sequence $(T_n)_{n=1}^{\infty}$ in $\mathcal{F}$ such that $T_n(x_n) > n$ for $n = 1, 2, 3, \ldots$ and also $T_n(x) > n$ for the limit element x. This yields a contradiction to the hypothesis of the theorem.

From (3) it follows that there exists a $x_1 \in B(\mathbb{0}, 1)$ and $T_1 \in \mathcal{F}$ such that $\|T_1(x_1)\|_2 > 1$. Since T_1 is bounded linear and hence continuous there exists $0 < r_1 < \frac{1}{2}$ such that $\|T_1(x)\|_2 > 1$ for all $x \in B(x_1, r_1)$ and $\overline{B(x_1, r_1)} \subset B(\mathbb{0}, 1)$.

In the same way it follows from (3) that there exists a $x_2 \in B(x_1, r_1)$ and $T_2 \in \mathcal{F}$ such that $\|T_2(x_2)\|_2 > 2$. Moreover since T_2 is bounded linear it follows that there exists $0 < r_2 < (\frac{1}{2})^2$ such that $\|T_2(x)\|_2 > 2$ for all $x \in B(x_2, r_2)$ and $\overline{B(x_2, r_2)} \subset B(x_1, r_1)$. Proceed inductively. We obtain a sequence $(x_n)_{n=1}^{\infty}$ in E_1 and a sequence $(T_n)_{n=1}^{\infty}$ in $\mathcal{F}$ and $(r_n)_{n=1}^{\infty}$ in $(0, \infty)$ such that for $n = 1, 2, 3, \ldots$

- $\|T_n x\|_2 > n$ for all $x \in B(x_n, r_n)$
- $\overline{B(x_n, r_n)} \subset B(x_{n-1}, r_{n-1})$
- $0 < r_n < (\frac{1}{2})^n$

We conclude that $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence in E_1 since for $n > m$

$$\|x_n - x_m\|_1 < r_m < (\frac{1}{2})^m \to 0 \text{ as } n, m \to \infty.$$

Since $(E_1, \|\cdot\|_1)$ is a Banach space the sequence $(x_n)_{n=1}^{\infty}$ converges. Call the limit x. Here $x \in \overline{B(x_n, r_n)} \subset B(x_{n-1}, r_{n-1})$ for all n and hence

$$\|T_{n-1}(x)\|_2 > n - 1 \;\; n = 2, 3, \ldots$$

and so

$$\sup_{T \in \mathcal{F}} \|T(x)\|_2 \geq \sup_{n=1,2,3,\ldots} \|T_n(x)\|_2 = \infty.$$

This is a contradiction to the hypothesis and the assumption (3) is false. The theorem is proved. □

The Banach-Steinhaus theorem will be used later several times in the course. Here we just give a corollary to the theorem.

Corollary 0.1. *Let E_1 be a Banach space and E_2 a normed space. Moreover let $(T_n)_{n=1}^{\infty}$ be a sequence in $\mathcal{B}(E_1, E_2)$ such that*

$$T(x) \equiv \lim_{n\to\infty} T_n(x) \text{ exists for all } x \in E_1.$$

Then

$$T \in \mathcal{B}(E_1, E_2).$$

To sketch the proof, note that T inherits the property of being linear from that for all T_n. Since $(T_n(x))_{n=1}^{\infty}$ converges in E_2 for all $x \in E_1$ it is a bounded sequence for all x and hence by the Banach-Steinhaus theorem $\sup_n \|T_n\| < \infty$. This implies that T is bounded liner mapping since for $x \in E_1$

$$\|T(x)\| = \|\lim_{n\to\infty} T_n(x)\| = \lim_{n\to\infty} \|T_n(x)\| \leq \sup_n \|T_n\| \, \|x\|.$$

Week 3 Exercise K 6.4:11 was taken as an introduction to the Banach's fixed point theorem, theorems 2.1 and 2.2 were stated and proved, exercises K 6.4:3,9 were solved, example on page 21 was discussed. Solutions to some exercises for week 1 and 2 were sketched. A proof of Picard's existence theorem for solutions to initial value problems is given here. It resembles the proof of the statement in the example on page 21.

Theorem 0.3. *Let $\|\cdot\|$ be any norm on $\mathbb{R}^n$ and let $T > 0$. Assume that*

$$\mathbb{g} \in C([0, T] \times \mathbb{R}^n; \mathbb{R}^n)$$

and that there exists a $\gamma > 0$ such that

$$\|\mathbb{g}(t, \mathbb{v}) - \mathbb{g}(t, \mathbb{w})\| \leq \gamma \|\mathbb{v} - \mathbb{w}\| \text{ for all } t \in [0, T] \text{ and } \mathbb{v}, \mathbb{w} \in \mathbb{R}^n.$$

Let $\mathbb{u}_0 \in \mathbb{R}^n$. Then the initial value problem

$$\begin{cases} \mathbb{u}'(t) = \mathbb{g}(t, \mathbb{u}(t)), & t \in [0, T] \\ \mathbb{u}(0) = \mathbb{u}_0 \end{cases} \tag{4}$$

has a unique solution $\mathbb{u} \in C([0, T]; \mathbb{R}^n)$.

We note that (4) is a system of 1:st order differential equations, that the interval $[0, T]$ can be replaced by any interval $[t_0, \tilde{T}]$ for any $t_0 < \tilde{T}$ and that[1]

$$\mathbb{u}'(t) = [u_1'(t) \; u_2'(t) \; \ldots \; u_n'(t)]^T.$$

A general n-order differential equation

$$u^{(n)}(t) = f(t, u(t), \ldots, u^{(n-1)})(t)$$

can be written as a system of 1:st order differential equations by setting

$$\begin{cases} u_1(t) = u(t) \\ u_2(t) = u'(t) \\ \ldots \\ u_n(t) = u^{(n-1)}(t) \end{cases}$$

[1] T in the expression below stands for taking the transpose of the matrix.

With

$$\mathbb{u}(t) = [u_1(t)\ u_2(t)\ \dots\ u_n(t)]^T$$

the equation takes the form

$$\mathbb{u}'(t) = A\mathbb{u}(t) + [0\,0\,\dots\,f(t, \mathbb{u}(t))]^T,$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & & \dots & \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

PROOF: (somewhat sketchy) The proof is done in three steps.

Step 1: We observe that if $\mathbb{u} \in C([0,T];\mathbb{R}^n)$ solves

$$\mathbb{u}(t) = \mathbb{u}_0 + \int_0^t \mathbb{g}(s, \mathbb{u}(s))\, ds, \quad t \in [0,T]$$

then $\mathbb{u} \in C^1([0,T];\mathbb{R}^n)$ and is a solution to the IVP (4), and conversely.

Step 2: We observe that $C([0,T];\mathbb{R}^n)$ is a Banach space with the norm

$$|||\cdot||| : C([0,T];\mathbb{R}^n) \ni \mathbb{v} \mapsto \sup_{t\in[0,T]} e^{-\gamma t}\|\mathbb{v}(t)\|.$$

The mapping

$$\mathbb{F} : C([0,T];\mathbb{R}^n) \to C([0,T];\mathbb{R}^n)$$

defined by

$$\mathbb{F}(\mathbb{v})(t) = \mathbb{u}_0 + \int_0^t \mathbb{g}(s, \mathbb{v}(s))\, ds, \quad t \in [0,T]$$

is a contraction with respect to the norm $|||\cdot|||$ since

$$(\mathbb{F}(\mathbb{v}) - \mathbb{F}(\mathbb{w}))(t) = \int_0^t e^{\gamma s} \cdot e^{-\gamma s} \cdot (\mathbb{g}(s, \mathbb{v}(s)) - \mathbb{g}(s, \mathbb{w}(s))\, ds$$

and hence

$$\|(\mathbb{F}(\mathbb{v}) - \mathbb{F}(\mathbb{w}))(t)\| \leq (*) \leq \int_0^t e^{\gamma s}\, ds \cdot \underbrace{\sup_{s\in[0,T]} e^{-\gamma s}\|\mathbb{g}(s, \mathbb{v}(s) - \mathbb{g}(s, \mathbb{w}(s))\|}_{\leq \gamma|||\mathbb{v}-\mathbb{w}|||} \leq e^{\gamma t}(1 - e^{-\gamma T})|||\mathbb{v} - \mathbb{w}|||.$$

Here we have used a generalization of the triangle inequality for norms at $(*)$ when moving the norm inside the integral. Banach's fixed point theorem implies that there exists a unique $\mathbb{u} \in C([0,T];\mathbb{R}^n)$ such that

$$\mathbb{u}(t) = \mathbb{F}(\mathbb{u})(t) = \mathbb{u}_0 + \int_0^t \mathbb{g}(s, \mathbb{u}(s))\, ds, \quad t \in [0,T].$$

Step 3: From Step 1 the conclusion of the theorem follows. $\square$

Week 4 Introduced inner product spaces with examples $(l^2, \langle \cdot, \cdot \rangle_{l^2})$ and $(C([0,1]), \langle \cdot, \cdot \rangle_{L^2})$. Stated and proved Cauchy-Schwarz inequality and that the induced norm $\|x\| = \sqrt{\langle x, x \rangle}$ is a norm that satisfies the parallelogram law. Used this to give examples of non-inner product spaces. Introduced the polarization identity and gave consequences. Defined orthogonality and ON-sequence. Stated and proved the Pythagorean theorem, Bessel's equality and inequality. Defined the notion of Hilbert space. Defined strong and weak convergence and relations for these notions. Stated and proved that a weakly converging sequence in a Hilbert space is bounded. Definition of orthogonal complement. Stated and showed that for any subset A in a Hilbert space $A^{\perp}$ is a closed subspace and that $A^{\perp\perp} = \overline{\mathrm{Span} A}$.

An application of Schaauder's fixed point theorem was treated. The notion of completion of a normed space was discussed in general and in the context of L^p-spaces. Basic properties of L^p-functions were stated.

Week 5 Riesz representation theorem was stated, proved and discussed. Gram-Schmidt orthogonalization procedure was given. It was shown that for an ON-sequence $(x_n)_{n=1}^{\infty}$ in a Hilbert space E,

$$\Sigma_{n=1}^{\infty} \alpha_n x_n \in E \text{ iff } (\alpha_n)_{n=1}^{\infty} \in l^2.$$

Definition of ON-basis. Proved that if $(x_n)_{n=1}^{\infty}$ is an ON-sequence in a Hilbert space $(E, \langle \cdot, \cdot \rangle)$ then the following statements are equivalent:

- $(x_n)_{n=1}^{\infty}$ is a complete ON-sequence
- $\mathrm{Span}\{x_n : n = 1, 2, 3, \ldots\}$ is dense in E
- Parseval's formula
- For every $x, y \in E$
$$\langle x, y \rangle = \Sigma_{n=1}^{\infty} \langle x, x_n \rangle \overline{\langle y, x_n \rangle}$$
- $\langle x, x_n \rangle = 0$ for all n implies $x = \mathbb{0}$

Defined the concept of separable Hilbert space and showed that such a space has a countable dense subset. Trigonometric Fourier series were mentioned.

Linear bounded operators on a Banach (e.g. Hilbert) space were defined. The standard example with an integral operator on $L^2([0,1])$ with continuous kernel function k was discussed. It was shown that the operator norm for the integral operator is bounded above by the L^2-norm of the function k. Composition of bounded linear operators A, B was defined and it was proved that

$$\|AB\| \leq \|A\| \, \|B\|$$

It was proven that there can be no bounded linear operators A, B on any normed space satisfying $AB - BA = I$, where $I(x) = x$ for all x.

Lax-Milgram's theorem was stated and proved.

The concept of adjoint operator to a bounded linear operator in a Hilbert space was discussed including elementary properties. The adjoint operator for our standard example was calculated. The notion of self-adjoint operator was introduced and it was stated and proved that for a self-adjoint operator A the operator norm is (also) given by

$$\|A\| = \sup_{\|x\|=1} |\langle A(x), x\rangle|$$

Proved that for $A \in \mathcal{B}(E, E)$ for Hilbert space E

1. $\mathcal{R}(A)^{\perp} = \mathcal{N}(A^*)$
2. $\overline{\mathcal{R}(A)} = \mathcal{N}(A^*)^{\perp}$

Week 6 Definition of compact (linear) operators on Banach/Hilbert spaces and basic properties for these, stated and proved that the compact linear operators on a Banach space form a closed subspace in the vector space of bounded linear operators on the Banach space with the operator norm. We showed that compact operators improves convergence in the Hilbert space setting (a weakly converging sequence is mapped by the compact operator to a strongly converging sequence). Moreover we showed that finite-rank operators are compact and that every compact operator in a Hilbert space can be approximated by finite-rank operators (not true for Banach spaces!). We also discussed that our "standard integral operator on $L^2([0, 1])$ with continuous kernel function" is compact. Fredholm Alternative was stated, also in the case of compact operators on a Banach space, and proved for compact operators on a Hilbert space. The notions of eigenvalue/eigenfunction was introduced and basic properties for these were discussed. It was stated and proved that a compact self-adjoint operator A has an eigenvalue λ with $|\lambda| = \|A\|$. Hilbert-Schmidt theorem/spectral theorem for compact self-adjoint operators was stated and proved. The spectrum for an bounded liner operator was defined and basic properties were discussed, e.g. the Neumann-series lemma. Properties for the spectrum for a compact opertor on a Banach/Hilbert space was discussed.

Week 7 Boundary value problems for ordinary differential equations were discussed. Method for calculating the Green's function was given. Application of Hilbert-Schmidt theorem to symmetric linear differential operators on intervals and with "sound" boundary conditions were given. The method of continuity (in spectral theory) was stated and proved. Orthogonal projection operators on Hilbert spaces were discussed. Finally an classical exercise involving ON-basis' was solved.

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1 Introduction

Functional analysis is one of the major fields of mathematics. It traces its history back to the nineteen century. From this period we find mathematicians like Fredholm (a swede) and Volterra. Many breakthroughs were made during the first part of the twentieth century. Names associated with these important years are Hilbert and Banach. We will indeed meet these names later! These notes will only serve as a complement to the textbook by Debnath and Mikusinski with the title *An Introduction to Hilbert Spaces with Applications*. The additional material has been added to the course previously and for the benefit of the students written down in the notes.

As a motivating example we discuss a problem coming from differential equations. Looking at the problem in hinsight we can easily identify many of the different notions that appear in course and also get a feeling for them as being "natural". So let us roll up are sleeves and start with the calculations.

Consider the differential equation

$$f''(x) + f(x) = g(x) \tag{5}$$

in the interval $0 \leq x \leq 1$ with the solution satisfying the boundary conditions

$$f(0) = 1, \; f'(0) = 0.$$

The problem itself is of no special interest and will only serve as a testing ground for our ideas. Here we first think of $g(x)$ as a given continuous function on the interval $x \in [0, 1]$. If $g = 0$ we know from our first calculus class that

$$f(x) = A \cos x + B \sin x$$

is a solution to the differential equation, where A, B are arbitrary constants. To treat the case with an arbitrary function $g(x)$ we apply the *method of variations of constants*. If you are not familiar with the method it does not matter since this is the only time when we will use it. Set

$$f(x) = A(x) \cos x + B(x) \sin x \tag{6}$$

and differentiate. We get

$$f'(x) = A'(x) \cos x + B'(x) \sin x - A(x) \sin x + B(x) \cos x.$$

Assume (and this is part of the method) that

$$A'(x) \cos x + B'(x) \sin x = 0, \; x \in [0, 1].$$

Differentiate once more. This gives

$$f''(x) = -A(x) \cos x - B(x) \sin x - A'(x) \sin x + B'(x) \cos x.$$

Hence (6) satisfies (5) if

$$-A'(x) \sin x + B'(x) \cos x = g(x), \; x \in [0, 1].$$

We now solve

$$\begin{cases} A'(x)\cos x + B'(x)\sin x = 0 \\ \\ -A'(x)\sin x + B'(x)\cos x = g(x) \end{cases}$$

This together with the boundary conditions gives us that

$$\begin{cases} A'(x) = -g(x)\sin x \\ \\ B'(x) = g(x)\cos x \\ \\ A(0) = 1 \ (= f(0)) \\ \\ B(0) = 0 \ (= f'(0)) \end{cases}$$

We conclude that

$$\begin{cases} A(x) = A(0) + \int_0^x A'(t)\,dt = 1 + \int_0^x (-g(t)\sin t)\,dt \\ \\ B(x) = B(0) + \int_0^x B'(t)\,dt = \int_0^x g(t)\cos t\,dt \end{cases}$$

which finally implies that

$$f(x) = \cos x + \int_0^x \sin(x-t)g(t)\,dt. \tag{7}$$

You can easily check that this function $f(x)$ satisfies the differential equation and the imposed boundary conditions. (7) is a reformulation of the differential equation with the boundary conditions.

To push things further consider the case with $g(x) = k(x)f(x),\ x \in [0,1]$. Here k is assumed to be a known continuous function on $[0,1]$. The solution formula above implies

$$f(x) = \cos x + \int_0^x \sin(x-t)k(t)f(t)\,dt. \tag{8}$$

Note that the function f appears on both sides. Here comes a main idea.

Pick any $f_0(x) \in C([0,1])$. $C([0,1])$ denotes the set of all continuous functions on $[0,1]$. Set

$$\begin{cases} f_1(x) = \cos x + \int_0^x \sin(x-t)k(t)f_0(t)\,dt \\ \\ f_2(x) = \cos x + \int_0^x \sin(x-t)k(t)f_1(t)\,dt \\ \\ \dots \end{cases}$$

i.e.

$$f_n(x) = \cos x + \int_0^x \sin(x-t)k(t)f_{n-1}(t)\,dt \quad n = 1, 2, 3, \dots$$

To simplify notations set $u(x) = \cos x$ and

$$Kv(x) = \int_0^x \sin(x-t)k(t)v(t)\,dt, \quad v \in C([0,1]),\ x \in [0,1].$$

Then equation (8) takes the form

$$f = u + Kf.$$

Now consider the sequence $(f_n)_{n=0}^{\infty}$ where

$$f_n = u + Kf_{n-1},\ n = 1, 2, 3, \dots$$

Dream: f_n "tends to" a continuous function f and Kf_n "tends to" Kf as $n \to \infty$. Here we have to make "tends to", which we denote by $\to$, precise. The dream is illustrated by the diagram

$$\begin{array}{ccccc} f_n & = & u & + & Kf_n \\ \downarrow & & & & \downarrow \\ f & = & u & + & Kf \end{array} \tag{9}$$

The limit function f will be a solution to our problem.

To proceed we recall some basic facts from first year calculus courses.

Definition 1.1. *We say that a sequence $(v_n)_{n=1}^{\infty}$ of continuous functions on $I = [0,1]$* **converges uniformly on** *I if*

$$\max_{x \in I} |v_n(x) - v_m(x)| \to 0, \ \text{ as } \ n, m \to \infty \tag{10}$$

i.e. if for all $\epsilon > 0$ there exists N such that

$$\max_{x \in I} |v_n(x) - v_m(x)| < \epsilon, \ \text{ for all } \ n, m \geq N. \tag{11}$$

Lemma 1.1. *Suppose that $(v_n)_{n=1}^{\infty}$ converges uniformly on I. Then there exists a continuous function v on I such that*

$$\max_{x \in I} |v_n(x) - v(x)| \to 0, \ \text{ as } \ n, m \to \infty.$$

Moreover we set $\|h\| = \max_{x \in I} |h(x)|$ for every $h \in C(I)$. Then (10) and (11) can be written as

$$\|v_n - v_m\| \to 0, \ \text{ as } \ n, m \to \infty$$

and

$$\|v - v_n\| \to 0, \ \text{ as } \ n \to \infty$$

Back to our problem above: The question is now whether

$$\|f_n - f_m\| \to 0, \ \text{ as } \ n, m \to \infty$$

or not? Does this depend on the choice of f_0?

To settle that question note that with notations from above

$$K(v + w) = Kv + Kw \ \text{ for all } \ v, w \in C(I),$$

where $v+w$ is the (continuous!) function that is defined by $(v+w)(x) = v(x)+w(x)$ for $x \in I$ (and also $K(v+w)$ continuous). Moreover we set

$$K^n v = K(K^{n-1}v), \text{ for all } v \in C(I) \text{ and } n = 2,3,4,\ldots$$

We now have

$$f_1 = u + Kf_0$$

$$f_2 = u + Kf_1 = u + K(u+Kf_0) = u + Ku + K^2 f_0$$

$$\ldots$$

$$f_n = u + Ku + K^2 u + \ldots + K^{n-1}u + K^n f_0$$

Assume $n > m$. We obtain

$$f_n - f_m = K^m u + \ldots + K^{n-1}u + K^n f_0 - K^m f_0.$$

From the triangle inequality for real numbers we get

$$\|v+w\| \leq \|v\| + \|w\|$$

We also have $\|-v\| = \|v\|$. This gives

$$\|f_n - f_m\| \leq \|K^m u\| + \ldots + \|K^{n-1}u\| + \|K^n f_0\| + \|-K^m f_0\| = \Sigma_{l=m}^{n-1}\|K^l u\| + \|K^n f_0\| + \|K^m f_0\|.$$

If

$$\Sigma_{l=1}^{\infty}\|K^l v\| < \infty \text{ for every } v \in C(I), \tag{12}$$

then

$$\|f_n - f_m\| \to 0, \text{ as } n,m \to \infty.$$

Asssume that (12) **holds for the moment**. Then we can conclude that

- there exists a $f \in C(I)$ such that $\|f_n - f\| \to 0$ as $n \to \infty$ (which we write $f_n \to f$)
- $\|Kf_n - Kf\| \to 0$ as $n \to \infty$ since:

  For $x \in [0,1]$

  $$|Kf_n(x) - Kf(x)| = |\int_0^x \sin(x-t)k(t)(f_n(t)-f(t))\,dt| \leq$$

  $$\leq \int_0^x |\sin(t-x)| \cdot |k(t)| \cdot |(f_n-f)(t)|\,dt \leq \|k\|\,\|f_n - f\|x$$

  so

  $$\|Kf_n - Kf\| \leq \|k\|\,\|f_n - f\| \to 0 \text{ as } n \to \infty.$$

  We have found yet another property of K namely

  $$\|Kv\| \leq M\|v\| \text{ for all } v \in C(I)$$

  (with $M = \|k\|$)

- The diagram (9) is proven to hold!

We can conclude, provided (12) holds, that the problem

$$\begin{cases} f'' + f = kf, \ x \in I \\ \\ f(0) = 1, \ f'(0) = 0 \end{cases} \tag{13}$$

reformulated as the integral equation $f(x) = \cos x + \int_0^x \sin(x-t)k(t)f(t)\,dt$ has a solution.

It remains to show that (12) holds. Fix a $v \in C(I)$. For $x \in I$ we get

$$|Kv(x)| = |\int_0^x \sin(x-t)k(t)v(t)\,dt| \le \int_0^x \|k\|\,\|v\|\,dt = \|k\|\,\|v\|x$$

$$|K^2v(x)| = |\int_0^x \sin(x-t)k(t)Kv(t)\,dt| \le \int_0^x \|k\||Kv(t)|\,dt \le$$

$$\le \int_0^x \|k\|^2\|v\|t\,dt = \|k\|^2\|v\| \cdot \tfrac{x^2}{2}$$

By induction we get

$$|K^n v(x)| \le \|k\|^n\|v\| \cdot \frac{x^n}{n!} \ \ n = 1, 2, 3, \ldots$$

and hence

$$\Sigma_{l=1}^{\infty}\|K^l v\| \le \Sigma_{l=1}^{\infty}\frac{\|k\|^l\,\|v\|}{l!} \le e^{\|k\|} \cdot \|v\| < \infty.$$

Claim: $f = u + Kf$ has a unique continuous solution f

Assume that there are two solutions $f, \tilde{f}$. Set $v = f - \tilde{f}$. Then it holds that

$$v = Kf - K\tilde{f} = K(f - \tilde{f}) = Kv$$

and so

$$v = Kv = K^2v = \ldots = K^l v \ \ l = 1, 2, 3, \ldots$$

But then

$$\Sigma_{l=1}^{\infty}\|K^l v\| < \infty$$

implies that $\|v\| = 0$, i.e. $v(x) = 0, \ x \in I$. Hence $f = \tilde{f}$ and the solution is unique.

Reconsidering the calculations above we have more ore less proved the following result with our bare hands.

Theorem 1.1. *Consider the integral equation*

$$f(x) = u(x) + \int_0^x k(x,t)f(t)\,dt, \ \ x \in [0,1] = I, \tag{14}$$

where $u \in C(I)$ and $k \in C(I \times I)$. Then there exists a unique $f \in C(I)$ satisfying (14).

With the same technology we can prove

Theorem 1.2. *Consider the integral equation*

$$f(x) = u(x) + \int_0^1 k(x,t)f(t)\,dt, \quad x \in [0,1] = I, \tag{15}$$

where $u \in C(I)$ *and* $k \in C(I \times I)$ *and with*

$$\max_{(x,t)\in I\times I} |k(x,t)| < 1.$$

Then there exists a unique $f \in C(I)$ *satisfying* (15).

The integral equation in the first theorem is called a *Volterra integral equation* and the one in the second theorem is called a *Fredholm integral equation.* We will come back to these integral equations later in the course.

It was indicated in the beginning that this introductory example would serve as a test bench for different notions that will appear in the course. A quick odyssey through these notions involves

vector space: $C([0,1])$ with the operations addition and multiplication by scalars, defined by

$$\begin{cases} (u+v)(x) = u(x) + v(x), & x \in [0,1] \\ (\lambda u)(x) = \lambda u(x), & x \in [0,1] \end{cases}$$

for $u, v \in C([0,1])$ and scalars λ, defines a vector space. Note that $u+v$ and λu are continuous functions.

norm on a vector space: $\|u\| = \max_{x\in I} |u(x)|$ defines a norm on the vector space $C([0,1])$. The norm gives a way to measure distance between the elements in the vector space. It also has properties (triangle inequality and scaling) related to the two operations (addition and multiplication by scalars) on the vector space. However a vector space can be equipped with many different norms. Once we have a norm we can talk about the notions of convergence and continuity (with respect to the particular norm). When we talk about real-valued continuous functions on the interval $[0,1]$ we consider $\mathbb{R}$ as a vector space (with the obvious definitions of addition and multiplication by scalars) and with the norm given by the absolute value $|\cdot|$ for real numbers.

Cauchy sequence on a normed space: $(f_n)_{n=1}^\infty$ defined in the example above is a Cauchy sequence in the vector space $C([0,1])$ with the max-norm. In general, a Cauchy sequence in a normed vector space is a sequence $(u_n)_{n=1}^\infty$ such that

$$\|u_n - u_m\| \to 0 \text{ as } n, m \to \infty.$$

Banach space: $C([0,1])$ with the max-norm is a Banach space since for every Cauchy sequence $(u_n)_{n=1}^\infty$ in $C([0,1])$ there exists a $u \in C([0,1])$ such that $\|u_n \to u\| \to 0$ as $n \to \infty$, in other words "every Cauchy sequence converges". This is the defining property for Banach spaces. We observe that the real numbers with the norm given by the absolute value is a Banach space!

linear mappings between vector spaces: K is a linear mapping from $C([0,1])$ into itself since

$$\begin{cases} K(u+v) = Ku + Kv \\ K(\lambda u) = \lambda Ku \end{cases}$$

hold for all $u, v \in C([0,1])$ and scalars λ.

bounded linear mappings: K is a bounded linear mapping if there exists a constant M such that

$$\|Ku\| \leq M\|u\|, \quad \text{for all} \;\; u \in C([0,1]).$$

operator norm:

$$\|K\| = \inf\{M > 0 : \|Ku\| \leq M\|u\|, \quad \text{for all} \;\; u \in C([0,1])\}$$

The bounded linear mappings between normed spaces form in a natural way a vector space in itself and the operator norm is a norm on this vector space. Note here that the norm signs $\|K\|$ and $\|Ku\|$ have different meanings, they are norms on different vetor spaces. When there is any risk of misinterpretation we will use different norm-signs.

fixed point theorems: To find or prove the existence of a solution to the problem $f = T(f)$, where T is a mapping (not necessarily linear as in our example) from a (closed) set F in a Banach space E into itself (i.e. F), we can sometimes proceed as follows: Pick a $f_0 \in F$ and then form the sequence of iterates $f_{n+1} = T(f_n)$, $n = 1, 2, 3, \ldots$ This gives a solution f as the limit element of $(f_n)_{n=0}^{\infty}$ if T satisfies

$$\|T(g) - T(h)\| \leq c\|g - h\| \;\; \text{all } g, h \in F$$

for some constant $0 < c < 1$. This is called Banach's fixed point theorem and will be used frequently.

Green's function: In our problem we have the linear differential operator

$$L = (\frac{d}{dx})^2 + 1,$$

acting on twice continuously differentiable functions on $I = [0,1]$, and with the homogeneous boundary conditions

$$f(0) = f'(1) = 0.$$

(The reason for the function $u(x) = \cos x$ that appears in the solution formula is to compensate for the inhomogeneity in the boundary conditions) We have that the solution to the problem

$$\begin{cases} Lf = h, \;\; \text{on } I \\ f(0) = f'(1) = 0 \end{cases}$$

can be written as

$$f(x) = \int_0^1 g(x,t)h(t)\,dt, \quad x \in I.$$

Here the Green's function g is given by $g(x,t) = \sin(x-t)\chi(x,t)$ where

$$\chi(x,t) = \begin{cases} 1 & x > t \\ 0 & x < t \end{cases}$$

2 Fixed point theory

2.1 Introduction

This section contains topics from nonlinear functional analysis. By this we mean that the mappings that appear are not assumed to be linear unless explicitly stated to be so.

Generally the problem is to solve equations of the form

$$T(u) = v,$$

where $T : X \to Y$ is a mapping between Banach spaces X and Y. Here $v \in Y$ is given and we look for solutions in X or some subset of X. For linear mappings T we can often find a formula for the inverse operator. The solution has to be uniquely defined in this case. To exemplify consider the boundary value problems

$$\begin{cases} u^{(n)} + a_{n-1}u^{(n-1)} + \ldots + a_1 u' + a_0 u = v, & \text{in } I \\ \qquad \text{homogeneous boundary values} & \text{on } \partial I \end{cases}$$

The solutions are obtained as convolutions of the Green's function for the problem with the right hand side v of the differential equation.

However if T is a nonlinear mapping then in general we can not find a formula representing the solution/solutions. This is also the case when $X = Y$. We can no longer prove the existence of a solution just by explicitly writing down the inverse operator, but we have rely on mapping properties of T to prove the existence of a solution. It might be the case that there are several solutions.

In connection with integral equations for instance we have $X = Y$ and the mapping T takes often the form

$$T(u) = u + G(u),$$

i.e. T is a perturbation of the identity mapping. The problem can be formulated as

$$u = H(u),$$

where $H(u) = v - G(u)$. We suppress the variable v and consider H as a function of u with v as a parameter. The problem to find a solution is then equivalent to find a fixed point of H, i.e. an element $u_0 \in X$ such that

$$u_0 = H(u_0).$$

We recall that if G is linear and small in the sense that the operator norm of G is less than 1 then the mapping T^{-1} is a welldefined bounded linear mapping and can be obtained as a Neumann series, see Section 4.

The fixed point results that will be discussed here are of two types. The first type deals with contractions and are referred to as metric fixed point theorems. One example of such a theorem is the Banach's fixed point theorem. The second type deals with compact mappings. Those are called topological fixed point theorems and are more involved. Names associated with such results are Brouwer and Schauder.

First let us consider a well-known example. Assume that

$$f : [0,1] \to [0,1]$$

is a continuous function. Then there exists a $x_0 \in [0,1]$ such that $f(x_0) = x_0$. This is a consequence of the theorem saying that every real-valued continuous function attains every intermediary value between any two given values and is based on the fact that

1. $[0,1]$ is a connected closed (i.e. a compact[2] and convex) subset in a Banach space, here $\mathbb{R}$, and that

2. f is a continuous function.

To prove the existence of a fixed point for f we usually define the function $g(x) = x - f(x)$ on the interval $[0,1]$ and observe that g is a continuous function satisfying $g(0) \le 0 \le g(1)$. We can then conclude that there is a $x_0 \in [0,1]$ such that $g(x_0) = 0$. This example can be considered as the 1-dimensional version of Brouwer fixed point theorem. One feature here is that the method is not constructive, i.e. the position of the fixed point is not given by the method. Nor does the method yield that the fixed point is unique, which indeed is sound since there can be any number of fixed points for f. To get some information on the position of one fixed point we can use the strategy of repeatedly bisecting intervals into pieces as follows: Assume that $g(0) < 0 < g(1)$, since otherwise we already have one fixed point, and consider the subintervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$. If $g(\frac{1}{2}) = 0$ we have one fixed point namely $x_0 = \frac{1}{2}$. If $g(\frac{1}{2}) > 0$ or $g(\frac{1}{2}) < 0$ we can apply the procedure to the the restriction of the function g to the subintervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ respectively. In this way we either find a fixed point as an end point of an interval or we find an infinite set of nested shrinking intervals that all contains a fixed point. For the later case we can for any $\epsilon > 0$ find an interval of length less than ϵ that contains a fixed point. We also note that this argument proves the intermediary value theorem provided we have that $\mathbb{R}$ is a complete normed space, i.e. a Banach space. Compare the argument above with the proof of Baire's theorem.

2.2 Banach's fixed point theorem

First we look at the problem to find a fixed point for a real-valued continuous function $f : \mathbb{R} \to \mathbb{R}$ in the spirit of Banach's fixed point theorem. We then need f to be a contraction meaning that there exists a positive real number c **less than** 1 such that for any pair x, y of points the distance between the images under f of these points is smaller by a factor c than the distance between the points x and y. In formulas this means

$$|f(x) - f(y)| \le c|x - y|$$

for arbitrary $x, y \in \mathbb{R}$. The conclusion from Banach fixed point theorem is that there is a unique fixed point for f. This can be found as follows: With notation from Banach's fixed point theorem fix any element $z \in \mathbb{R}$ and then form the sequence $(T^n(z))_{n=1}^{\infty}$. T^n denotes the

[2] cf. $g : (0,1) \to (0,1)$ with $g(x) = \frac{x}{2}$.

operator obtained by composing T with itself n times, i.e. $T^n = \underbrace{T \circ T \circ \ldots \circ T}_{n \text{ elements}}$. The sequence is converging geometrically with the fixed point as the limit point.

We first state and prove some general observations.

Lemma 2.1. *Let T be a continuous mapping on a Banach space X. Then the following statements hold true:*

1. *If there exist $x, y \in X$ such that*
$$\lim_{n\to\infty} T^n(x) = y$$
then y is a fixed point for T, i.e. $T(y) = y$.

2. *If $T(X)$ is a compact set in X and for each $\epsilon > 0$ there exists a $x_\epsilon \in X$ such that*
$$\|T(x_\epsilon) - x_\epsilon\| < \epsilon$$
then T has a fixed point.

Proof. Set $y_n = T^n(x)$, $n = 1, 2, \ldots$. If T is a continuous mapping then

$$T(y) = T(\lim_{n\to\infty} y_n) = \lim_{n\to\infty} T(y_n) = \lim_{n\to\infty} y_{n+1} = y,$$

which proves the first statement.

Assume that the assumptions of 2) are satisfied. Then for $n = 1, 2, \ldots$ there are $x_n \in X$ such that

$$\|T(x_n) - x_n\| < \frac{1}{n}. \tag{16}$$

$T(X)$ is a compact set which implies that there exits a convergent subsequence $(T(x_{n_k}))_{k=1}^\infty$ of $(T(x_n))_{n=1}^\infty$. Call the limit point x. Then x is a fixed point for T since also the sequence $(x_{n_k})_{k=1}^\infty$ converges to x according to (16) and T is continuous. $\square$

We now formulate one of the main theorems.

Theorem 2.1 (Banach's fixed point theorem)**.** *Let T be a contraction on a Banach space X. Then T has a unique fixed point.*

Proof. Fix an arbitrary element $z \in X$ and consider the sequence

$$(T^n(z))_{n=1}^\infty.$$

Set $z_n = T^n(z)$ for $n = 1, 2, \ldots$. We note that

$$\|z_n - z_m\| \leq \|z_n - z_{n-1}\| + \ldots + \|z_{m+1} - z_m\| =$$

$$= \|T(z_{n-1}) - T(z_{n-2})\| + \ldots + \|T(z_m) - T(z_{m-1})\| \leq$$

$$\leq c\|z_{n-1} - z_{n-2}\| + \ldots + c\|z_m - z_{m-1}\| \leq \ldots \leq$$

$$\leq (c^{n-1} + c^{n-2} + \ldots c^m)\|z_1 - z\| \leq \frac{c^m}{1-c}\|z_1 - z\|,$$

where we (without loss of generality) have assumed $n > m \geq 1$. This yields $\|z_n - z_m\| \to 0$ as $n, m \to \infty$ and hence $(z_n)_{n=1}^{\infty}$ is a Cauchy sequence. Since X is a Banach space the sequence converges, i.e. there is a $x_0 \in X$ such that $z_n \to x_0$ as $n \to \infty$. x_0 will be a fixed point for T since

$$\|T(x_0) - x_0\| \leq \|T(x_0) - T(z_n)\| + \|z_{n+1} - x_0\| \leq c\|x_0 - z_n\| + \|z_{n+1} - x_0\|$$

where the LHS is independent of n and the RHS tends to 0 as $n \to \infty$. The uniqueness follows from the contraction property for T. If $x_0 \neq y_0$ both are fixed points of T then we get

$$\|x_0 - y_0\| = \|T(x_0) - T(y_0)\| \leq c\|x_0 - y_0\| < \|x_0 - y_0\|$$

which results in a contradiction. □

From the proof we see that

1. the sequence $(T^n(z))_{n=1}^{\infty}$ converges to the unique fixed point **independently** of the choice of z.

2. for an arbitrary element $x \in X$ we have

$$\|x - x_0\| \leq \frac{1}{1-c}\|x - T(x)\|,$$

   where x_0 denotes the fixed point of T, since

$$\|x - x_0\| \leq \|x - T(x)\| + \|T(x) - T(x_0)\| \leq \|x - T(x)\| + c\|x - x_0\|.$$

Banach's fixed point theorem can be generalized in the following way.

Theorem 2.2. *Let T be a mapping on a Banach space X such that T^N is a contraction on X for some positive integer N. Then T has a unique fixed point.*

It is not necessary to assume that T is continuous.

Proof. Banach's fixed point theorem implies that there exists a unique fixed point for T^N. Call this element x_0. Now just note that

$$\|T(x_0) - x_0\| = \|T^N(T(x_0)) - T^N(x_0)\| \leq c\|T(x_0) - x_0\|$$

implies that $T(x_0) = x_0$ since $0 < c < 1$. The uniqueness is clear since a fixed point for T is also a fixed point for T^N. $\square$

Note that the conclusion of the previous theorem remains true if $T : F \to F$, where F is a closed set in the Banach space X, and T^N is a contraction for some positive integer N. Note that there is no assumption on F to be compact and/or convex.

Our next result shows that the fixed point depends continuously on a parameter if the mapping T also depends continuously on the parameter. More precisely we have

Theorem 2.3. *Let X be a Banach space and Y a normed space. Let $T : X \times Y \to X$ be a continuous mapping. Assume that T is a contraction on X uniformly in Y, that is, there is a $c < 1$ such that*

$$\|T(x_1, y) - T(x_2, y)\| \leq c\|x_1 - x_2\| \text{ for all } x_1, x_2 \in X, y \in Y.$$

Then for every fixed $y \in Y$, the mapping $x \mapsto T(x, y)$ has a unique fixed point $g(y) \in X$ and the mapping $y \mapsto g(y)$ is continuous from Y to X.

Notice that if $T : X \times Y \to X$ is continuous in Y and is a contraction on X uniformly in Y then T is continuous on $X \times Y$.

Proof. From Banach's fixed point theorem it follows that $g(y)$ is uniquely defined for all $y \in Y$. It remains to prove the continuity of g. For $y, \bar{y} \in Y$ we have

$$\|g(y) - g(\bar{y})\| = \|T(g(y), y) - T(g(\bar{y}), \bar{y})\| \leq$$

$$\leq \|T(g(y), y) - T(g(\bar{y}), y)\| + \|T(g(\bar{y}), y) - T(g(\bar{y}), \bar{y})\| \leq$$

$$\leq c\|g(y) - g(\bar{y})\| + \|T(g(\bar{y}), y) - T(g(\bar{y}), \bar{y})\|$$

which implies that

$$\|g(y) - g(\bar{y})\| \leq \frac{1}{1-c}\|T(g(\bar{y}), y) - T(g(\bar{y}), \bar{y})\|.$$

The RHS tends to 0 as $y \to \bar{y}$ and the continuity of g is proven. $\square$

Before we turn to some examples we squeeze in a generalization of Banach's fixed point theorem due to Boyd-Wong.

Theorem 2.4. *Let X be a Banach space and $T : X \to X$. Assume there exists a continuous function $\phi : [0, \infty) \to [0, \infty)$ such that $\phi(r) < r$ if $0 < r$ and*

$$\|T(x) - T(y)\| \leq \phi(\|x - y\|) \text{ for all } x, y \in X.$$

Then T has a unique fixed point $\bar{x}$. Moreover for any $x_0 \in X$ the sequence $(T^n(x_0))_{n=1}^{\infty}$ converges to $\bar{x}$.

Note that Banach's fixed point theorem corresponds to $\phi(r) = cr$ for $0 \leq r$.

Proof. The uniqueness of the fixed point is obvious. To prove the existence fix a $x_0 \in X$. We will prove the that $(T^n(x_0))_{n=1}^{\infty}$ is a Cauchy sequence. For $n = 1, 2, 3, \ldots$ set $x_{n+1} = T(x_n)$ and

$$a_n = \|x_n - x_{n-1}\|.$$

It is clear that $a_{n+1} \leq \phi(a_n)$ so a_n converges monotonically to some $a \geq 0$. From the continuity of ϕ we get $a \leq \phi(a)$ and hence $a = 0$. If $(x_n)_{n=1}^{\infty}$ is not a Cauchy sequence then there exists an $\epsilon > 0$ and integers $m_k > n_k \geq k$ for every positive integer k such that

$$b_k \equiv \|x_{m_k} - x_{n_k}\| \geq \epsilon \ \text{ all } k = 1, 2, 3, \ldots$$

In addition by choosing the smallest possible m_k we may assume

$$\|x_{m_k - 1} - x_{n_k}\| < \epsilon.$$

Therefore

$$\epsilon \leq b_k \leq \|x_{m_k} - x_{m_k-1}\| + \|x_{m_k-1} - x_{n_k}\| < a_{m_k} + \epsilon$$

implying that $b_k \to \epsilon$ as $k \to \infty$. Moreover

$$b_k \leq \|x_{m_k} - x_{m_k+1}\| + \|x_{m_k+1} - x_{n_k+1}\| + \|x_{n_k+1} - x_{n_k}\| \leq a_{m_k+1} + \phi(b_k) + a_{n_k+1}$$

and taking the limit as $k \to \infty$ we get $\epsilon \leq \phi(\epsilon)$. Contradiction! Hence $(T^n(x_0))_{n=1}^{\infty}$ is a Cauchy sequence. $\square$

We end this section by two examples. For the first one compare with the introductory example in the first section.

Example: Let $K(x, y)$ be a continuous real-valued function for $0 \leq y \leq x \leq 1$ and let $v(x)$ be a continuous real function for $0 \leq x \leq 1$. Then there is a unique continuous real function $z(x)$ such that

$$z(x) = v(x) + \int_0^x K(x, y) z(y)\, dy, \quad 0 \leq x \leq 1.$$

To prove this we consider the Banach space $C([0, 1])$ with the sup-norm and define the integral operator $L : C([0, 1]) \to C([0, 1])$ by

$$Lz(x) = \int_0^x K(x, y) z(y)\, dy.$$

Clearly L^n will be an integral operator on $C([0, 1])$ given by a kernel function $K_n(x, y)$. To find this function set $K_1(x, y) = K(x, y)$ and assume that $K_n(x, y)$ is known. Then we obtain

$$(L^{n+1}z)(x) = \int_0^x K(x, t)(L^n z)(t)\, dt = \int_0^x K(x, t) \int_0^t K_n(t, y) z(y)\, dy\, dt =$$

$$= \int_0^x \left(\int_y^x K(x, t) K_n(t, y)\, dt\right) z(y)\, dy = \int_0^x K_{n+1}(x, y) z(y)\, dy.$$

Hence

$$K_{n+1}(x,y) = \int_y^x K(x,t)K_n(t,y)\,dt, \quad 0 \le y \le x \le 1.$$

The function $K(x,y)$ is continuous on the closed set $\{(x,y) : 0 \le y \le x \le 1\}$ and so it is bounded, say

$$|K(x,y)| \le M$$

for all $0 \le y \le x \le 1$. Then again by induction we see that

$$|K_n(x,y)| \le \frac{M^n|x-y|^{n-1}}{(n-1)!}$$

for all $0 \le y \le x \le 1$. Indeed if this holds for n then for $0 \le y \le x \le 1$

$$|K_{n+1}(x,y)| \le \int_y^x M \frac{M^n|t-y|^{n-1}}{(n-1)!}\,dt = \frac{M^{n+1}|x-y|^n}{n!}.$$

Hence if N is sufficiently large we have

$$|K_N(x,y)| \le \frac{1}{2}$$

for $0 \le y \le x \le 1$ and so

$$|(L^N z)(x)| \le \int_0^x |K_N(x,y)|\,|z(y)|\,dy \le \frac{1}{2}\|z\|,$$

i.e.

$$\|L^N\| \le \frac{1}{2}.$$

We now define $T : C([0,1]) \to C([0,1])$ by $Tz = v + Lz$. This gives

$$T^N z = (\Sigma_{k=0}^{N-1} L^k)v + L^N z,$$

which yields that T^N is a contraction on $C([0,1])$. By Theorem 2.2 the mapping T has a unique fixed point.

Example: Let $K(x,y)$ and $f(y,z)$ be continuous real-valued functions for $0 \le x, y \le 1$ and $z \in \mathbb{R}$. Moreover let $v(x)$ be a continuous real function for $0 \le x \le 1$. Assume that

$$|f(y,z_1) - f(y,z_2)| \le N|z_1 - z_2|$$

for all $0 \le y \le 1$ and $z_1, z_2 \in \mathbb{R}$ and some $N > 0$. Our claim is that there exists a unique continuous function $z(x)$ on $0 \le x \le 1$ such that

$$z(x) = v(x) + \int_0^x K(x,y)f(y,z(y))\,dy.$$

As above we define $L : C([0,1]) \to C([0,1])$ by

$$Lz(x) = \int_0^x K(x,y)f(y,z(y))\,dy$$

and show that the map $T : C([0,1]) \to C([0,1])$, given by

$$T(z) = v + Lz$$

has a unique fixed point. Here comes a nice trick! For $a > 0$ we introduce a new norm $\|\cdot\|_a$ on $C([0,1])$:

$$\|z\|_a = \int_0^1 e^{-ay}|z(y)|\,dy.$$

Then $\|\cdot\|_a$ is indeed a norm on $C([0,1])$ which is equivalent to the L^1 norm. Set $X_a = (C([0,1]), \|\cdot\|_a)$ and let $\tilde{X}_a$ be the completion of X_a. Clearly $\tilde{X}_a$ is the vector space $L^1([0,1])$ with the norm $\|\cdot\|_a$, and L extends to a map $\tilde{L} : \tilde{X}_a \to \tilde{X}_a$ given by the formula for L. Furthermore with

$$M = \max_{0 \le x,y \le 1} |K(x,y)|$$

we have for $z_1, z_2 \in \tilde{X}_a$

$$\|\tilde{L}z_1 - \tilde{L}z_2\|_a = \int_0^1 e^{-ay}|\int_0^y K(y,t)(f(t,z_1(t)) - f(t,z_2(t)))\,dt|\,dy \le$$

$$\le MN \int_0^1 \int_0^y e^{-ay}|z_1(t) - z_2(t)|\,dt\,dy = MN \int_0^1 \int_t^1 e^{-ay}|z_1(t) - z_2(t)|\,dy\,dt =$$

$$= MN \int_0^1 \frac{e^{-at} - e^{-a}}{a}|z_1(t) - z_2(t)|\,dt \le \frac{MN}{a}\|z_1 - z_2\|_a.$$

This shows that for $a > MN$ the map

$$\tilde{L} : \tilde{X}_a \to \tilde{X}_a$$

is a contraction and so is $\tilde{T} = v + \tilde{L}$. It easily follows that $\tilde{T}$ maps $\tilde{X}_a$ into X_a, so the unique fixed point belongs to $C([0,1])$, and is also the unique fixed point for T.

Another version of the trick above is to equip $C([0,1])$ with the norm

$$|z|_a = \sup_{x \in [0,1]} |e^{-ax}z(x)|$$

with a large enough, which is equivalent to the standard sup-norm on $C([0,1])$. The reader is asked to check that the calculations above go through, i.e. L will be a contraction in $(C([0,1]), |\cdot|_a)$. An advantage here is that we do not need to consider any completion $\tilde{X}_a$.

2.3 Brouwer and Schauder fixed point theorems

We begin by formulating Brouwer's fixed point theorem.

Theorem 2.5 (Brouwer's fixed point theorem)**.** *Assume that K is a compact convex subset of $\mathbb{R}^n$ and that $T : K \to K$ is a continuous mapping. Then T has a fixed point in K.*

Observe that it does not follow from Brouwer fixed point theorem that the fixed point is unique. Consider for instance the identity operator on a compact convex set K in $\mathbb{R}^n$ for which every $x \in K$ is a fixed point.

Example 1: Take a street map for Goteborg and place it on the floor of a lecture room at Chalmers, say room MVF31. Then there will be a point on the map that coincides with the corresponding point in Goteborg. This follows from both Banach's fixed point theorem and Brouwer's fixed point theorem, where the former theorem also gives that the point is unique. Prove this to yourself!

Example 2: Let T_α denote the rotation α degrees around the center for a closed disc K of radius 1. Then Brouwer's fixed point theorem gives the existence of a fixed point for T_α (of course it is overkill to use a fixed point theorem to see that) while Banach's fixed point theorem cannot be applied directly[3] since T_α is not a contraction. It is obvious that the center of K is a fixed point but Brouwer's fixed point theorem also tells us that it is not possible to compose the rotation with a continuous deformation of the disc into itself in such a way that the composed mapping has no fixed point.

We note that

- (generalization of Brouwer's fixed point theorem): If there exists a homeomorphism, i.e. a continuous bijection with continuous inverse, between a compact convex set K in $\mathbb{R}^n$ and a set $\tilde{K}$, call the homeomorphism φ, and $\tilde{T} : \tilde{K} \to \tilde{K}$ is a continuous mapping then $\tilde{T}$ has a fixed point. To see this consider the mapping $T = \varphi^{-1} \circ \tilde{T} \circ \varphi$.
  Exercise: Prove that $\tilde{T}$ has a fixed point.
- it is enough to prove Brouwer fixed point theorem in the case $K = \overline{B(0,1)}$, where $B(a,r) = \{x \in \mathbb{R}^n : \|x - a\| < r\}$.

There are many proofs for Brouwer's fixed point theorem, both analytical, topological and also combinatorial. One starting point for a proof could be the following. Assume that $K = \overline{B(0,1)}$ and that T has no fixed point. Define the mapping $A : \overline{B(0,1)} \to \overline{B(0,1)}$ as follows: For every inner point x in $\overline{B(0,1)}$ let $\tilde{x}$ denote the point on the boundary $\partial B(0,1)$ that is the intersection of the ray from $T(x)$ through x and the boundary $\partial B(0,1)$. The ray is always well-defined since T has no fixed point. Now set

$$A(x) = \begin{cases} \tilde{x} & \text{if } \ x \in B(0,1) \\ x & \text{if } \ x \in \partial B(0,1) \end{cases}$$

Then A is a continuous mapping from $\overline{B(0,1)}$ into $\partial B(0,1)$ (verify this!) such that $A|_{\partial B(0,1)} = I|_{\partial B(0,1)}$. The challenge to show that T has no fixed point is now reformulated as to show that there is no continuous mapping $A : \overline{B(0,1)} \to \partial B(0,1)$ such that $A|_{\partial B(0,1)} = I|_{\partial B(0,1)}$. The

[3] Assume that the disc has its center at the origin in $\mathbb{R}^n$. Apply Banach's fixed point theorem to the operators $T_n = (1 - \frac{1}{n})T, \ n = 1, 2, \ldots$. We obtain a sequence of fixed points x_n to T_n such that

$$\|T(x_n) - x_n\| \le \frac{1}{n}, \ \ n = 1, 2, 3, \ldots.$$

The result follows from Theorem 1.1 above.

fact that there is no such mapping is deep but never the less intuitively obvious. Consider, for $n = 2$, an elastic membrane fixed on a circular frame. The existence of a mapping A implies that it should be possible to deform the membrane continuously in such a way that it in the end coincides with the frame without being fractured. For fixed $x \in B(0, 1)$ the mapping

$$t \mapsto (1 - t)x + tA(x),\ t \in [0, 1]$$

describes how this point on the membrane is moved from x at $t = 0$ to $A(x) \in \partial B(0, 1)$ at $t = 1$, under the deformation. Do not forget that the membrane should be fixed at the frame!!!

A beautiful proof based on Sperner's lemma will be indicated in the Exercises, see Section 6.

We present Perron's theorem as an application of Brouwer's fixed point theorem. Schauder's fixed point theorem will be applied in the context of nonlinear differential/integral equations to prove the existence of solutions.

Theorem 2.6 (Perron's theorem)**.** *Let A be a real $n \times n$–matrix with positive entries. Then there exists a positive eigenvalue for the linear mapping given by the matrix A, with an eigenvector with positive entries*

To prove Perron's theorem let K denote the set

$$\{(x_1, \ldots, x_n) : x_i \geq 0 \ \text{ all } i,\ \Sigma_{i=1}^n x_i = 1\}$$

and define $T(x) = Ax/\|Ax\|_{l^1}$ for $x \in K$. Apply Brouwer's fixed point theorem.

In a finite-dimensional normed space compactness is equivalent to closedness and boundedness. This is not the case in an infinite-dimensional normed space. The following example due to Kakutani should be compared to the next fixed point theorem due to Schauder.

Example: Let B denote the closed unit ball in $l^2(\mathbb{Z})$, where $l^2(\mathbb{Z})$ consists of all elements $\mathbb{x} = (\ldots, x_{-1}, x_0, x_1 \ldots)$ such that $\|\mathbb{x}\| = (\Sigma_{n=-\infty}^{\infty}|x_n|^2)^{\frac{1}{2}} < \infty$. It is clear that B is convex and bounded. Let $\mathbb{z}$ be the element in $l^2(\mathbb{Z})$ that satisfies $z_0 = 1$ and $z_n = 0$ for $n \neq 0$ and let S denote the shift operator defined by $(S(\mathbb{x}))_n = x_{n-1}$ for $n \in \mathbb{Z}$. Set

$$T : l^2(\mathbb{Z}) \to l^2(\mathbb{Z}),$$

where

$$T(\mathbb{x}) = S(\mathbb{x}) + (1 - \|\mathbb{x}\|)\mathbb{z}.$$

For $\mathbb{x} \in B$ we have

$$\|T(\mathbb{x})\| \leq \|S(\mathbb{x})\| + (1 - \|\mathbb{x}\|) = 1,$$

i.e. $T(\mathbb{x}) \in B$. But T has no fixed point in B since

$$(T(\mathbb{x}))_n = x_{n-1},\ n \neq 0$$

and

$$(T(\mathbb{x}))_0 = x_{-1} + (1 - \|\mathbb{x}\|),$$

which implies that $x_0 = x_1 = \ldots = x_n = \ldots$ and $x_{-1} = x_{-2} = \ldots = x_{-n} = \ldots$. This yields a contradiction since $\mathbb{x} \in l^2(\mathbb{Z})$.

From this example we see that a generalization of Brouwer's fixed point theorem to infinite-dimensional spaces should have the assumption that $T(K)$ is a compact set. We next formulate two versions of Schauder's fixed point theorem.

Theorem 2.7 (Schauder's fixed point theorem). *Assume that K is a convex compact set in a Banach space X and that $T : K \to K$ is a continuous mapping. Then T has a fixed point.*

For applications the following generalization proves to be useful.

Theorem 2.8 (generalization of Schauder's fixed point theorem). *Let F be a closed convex set in a Banach space X and assume that $T : F \to F$ is a continuous mapping such that $T(F)$ is a relatively compact subset of F. Then T has a fixed point.*

We recall that a set $K_1 \subset X$ is compact[4] if every sequence in K_1 has a convergent subsequence in K_1. Moreover we say that $K_2 \subset X$ is relatively compact if every sequence in K_2 has a subsequence that converges in X. The limit element of the converging sequence belongs to $\overline{K_2}$. The set K_2 being relatively compact implies that $\overline{K_2}$ is a compact set. Also an arbitrary subset of a compact set is relatively compact.

To prove Schauder's fixed point theorem we will make use of some new concepts and facts for compact sets. We say that the convex hull of a set F, denoted by $\operatorname{co} F$, is the set defined by

$$\bigcap_{F \subset H, H \text{ convex}} H.$$

By a convex combination of the elements $x_1, x_2, \ldots, x_n$ we mean a linear combination $\Sigma_{i=1}^n \lambda_i x_n$, where all $\lambda_i \geq 0$ and $\Sigma_{i=1}^n \lambda_i = 1$. An ϵ–net is a subset F_ϵ of F with the property that for each $x \in F$ there exists a $y \in F_\epsilon$ such that $\|x - y\| < \epsilon$.

Proposition 2.1. *The following statements are true:*

1. *A set F is relatively compact iff for each $\epsilon > 0$ there exists a finite ϵ–net.*
2. *A set K is compact iff it is closed and for every $\epsilon > 0$ there exists a finite ϵ–net.*
3. *The set $\operatorname{co} F$ is the same as the set of all convex combination of finitely many elements in F.*
4. *The convex hull of a finite set is compact.*
5. *K compact set implies that $\overline{\operatorname{co} K}$ is compact.*

The proof is left as an exercise. Here the last statement is not so easy to establish.

The proof of Schauder's fixed point theorem is quite easy to prove compared to Brouwer's fixed point theorem. Actually by an approximation procedure one can apply Brouwer's fixed point theorem to get the result in Schauder's theorem.

[4] This definition of compactness and relative compactness is sometimes referred to as sequential compactness and sequential relatively compactness in the literature. The words compactness and relatively compactness are then reserved to mean the following: A set K in a normed space is called compact if for each open cover of K there is a finite subcover. An open cover of K is a collection of open sets O_λ, $\lambda \in \Lambda$, whose union contains K as a subset. A finite subcover is a finite subset of $\{O_\lambda\}_{\lambda \in \Lambda}$ whose union also contains the set K. It can be shown that for metric spaces X the notions sequentially compact and compact are equivalent.

Proof. (of the Schauder theorems) The second Schauder theorem is a consequence of the first one. To see this assume that the hypothesis of the second theorem are satisfied. It then follows that the closed hull $\overline{R}$ of $R = T(F)$ is compact and so also $\overline{\operatorname{co}\overline{R}}$. Set $K = \overline{\operatorname{co}\overline{R}}$. We see that $K \subset F$ since F is closed and convex. Moreover $T : K \to K$ is continuous. Hence the second theorem follows from the first theorem.

It remains to prove the first theorem. This will be done by approximating the compact set K by compact sets K_n, $n = 1, 2, \dots$ in finite-dimensional spaces and approximating the mapping T by continuous mappings $T_n : K_n \to K_n$, where the approximation becomes better and better for larger n. Brouwer's fixed point theorem gives a sequence of fixed points (x_n) for the sequence (T_n), from which a converging subsequence of points (x_{n_k}) can be extracted. The limit element of this sequence will be a fixed point for T.

For every positive integer n we define mappings P_n, called Schauder projections, as follows: The compactness of K implies that there are finitely many elements $x_1, \dots, x_k \in K$ such that

$$K \subset \bigcup_{i=1}^{k} B(x_i, \frac{1}{n}).$$

Set

$$f_i(x) = \max(0, \frac{1}{n} - \|x - x_i\|),\ i = 1, \dots, k.$$

For every $x \in K$ there exists an i such that $f_i(x) > 0$. This implies that $\Sigma_{i=1}^{k} f_i(x) > 0$ for all $x \in K$. Set $K_n = \overline{\operatorname{co}\{x_1, \dots, x_k\}}$ and

$$P_n(x) = \frac{\Sigma_{i=1}^{k} f_i(x) x_i}{\Sigma_{i=1}^{k} f_i(x)},\ \ x \in K.$$

Finally we define $T_n = P_n T|_{K_n}$. We can now apply Brouwer's theorem to every mapping

$$T_n : K_n \to K_n,\ \ n = 1, 2, \dots$$

This yields a sequence of fixed points $\tilde{x}_n$ for T_n, i.e.

$$P_n T(\tilde{x}_n) = \tilde{x}_n,$$

and hence we get

$$\|T(\tilde{x}_n) - \tilde{x}_n\| < \frac{1}{n}.$$

Schauder's theorem now follows from Lemma 1.1. □

2.4 Continuity and applications

To apply the fixed point theorems above some results for continuous functions will often be used.

Theorem 2.9. *Assume that T is a continuous mapping between two Banach spaces X and Y. Then the following statements are true:*

1. *If K is a compact set in X then $T(K)$ is a compact set in Y.*

2. *If $Y = \mathbb{R}$ then T attains its maximum and its minimum on every compact set K in X, i.e. there are $x_0, x_1 \in K$ such that*
$$\sup_{x \in K} f(x) = T(x_0) = \max_{x \in K} T(x)$$
and
$$\inf_{x \in K} T(x) = T(x_1) = \min_{x \in K} T(x).$$

3. *T is uniformly continuous on every compact set in X.*

The different notions of continuity that will be used are the following:
Let $T : X \to Y$ be a mapping between two Banach spaces. Then T is called

continuous if for every $x \in X$ and each $\epsilon > 0$ there exists a $\delta = \delta(x, \epsilon) > 0$ such that for every $y \in X$
$$\|y - x\|_X < \delta \Rightarrow \|T(y) - T(x)\|_Y < \epsilon.$$

uniformly continuous on A, where $A \subset X$, if for every $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that for every $x, y \in A$ we have
$$\|y - x\|_X < \delta \Rightarrow \|T(y) - T(x)\|_Y < \epsilon.$$

If $T_\lambda : X \to Y$, $\lambda \in \Lambda$ is a set of mappings (finitely many or infinitely many) between two Banach spaces then these are called

equicontinuous on A, where $A \subset X$, if for every $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that for every pair of elements $x, y \in A$ and every $\lambda \in \Lambda$ we have
$$\|y - x\|_X < \delta \Rightarrow \|T_\lambda(y) - T_\lambda(x)\|_Y < \epsilon.$$

Proof. (of Theorem 2.8) To prove statement 1) let $T : X \to Y$ be a continuous mapping and K a compact set in X. Pick an arbitrary sequence $(y_n) \subset T(K)$. Then there exists a sequence (x_n) in K such that $T(x_n) = y_n$ for all n. The sequence (x_n) might not be uniquely determined since T is not assumed to be injective. But since K is a compact set there exists a convergent subsequence (x_{n_k}) of (x_n) in K, i.e. there is an element $x \in K$ such that $x_{n_k} \to x$ as $k \to \infty$. Moreover since T is continuous we have
$$x_{n_k} \to x \Rightarrow y_{n_k} = T(x_{n_k}) \to T(x) \in T(K).$$
This proves 1).

The proof of statement 2) is left as an exercise.

To prove statement 3) assume that K is a compact set of X and that $T : X \to Y$ is continuous. Moreover assume that T is not uniformly continuous on K. Then there exists an $\epsilon > 0$ such that for all positive integers n there are points $x_n, y_n \in K$ such that
$$\|y_n - x_n\|_X < \frac{1}{n} \tag{17}$$

and

$$\|T(y_n) - T(x_n)\|_Y \geq \epsilon. \tag{18}$$

But K is a compact set and so there exists a convergent subsequence (x_{n_k}) of (x_n), i.e. for some $x \in K$ we have $x_{n_k} \to x$. From (17) it follows that $y_{n_k} \to x$ since we have

$$\|y_{n_k} - x\|_X \leq \|y_{n_k} - x_{n_k}\|_X + \|x_{n_k} - x\|_X.$$

Moreover T is continuous and so $T(x_{n_k}) \to T(x)$ and $T(y_{n_k}) \to T(x)$. This gives a contradiction of (18). The statement 3) is proved. □

The Banach spaces that will be used in applications are $C(A)$ and $L^p(A)$, $1 \leq p < \infty$. Here A stands for different subsets of $\mathbb{R}^n$ for $n \geq 1$. Of course the norms should be the proper ones e.g. the sup-norm should be used for $C(A)$. We tacitly understand that the proper norm is used unless something else is stated. In the context of Schauder's fixed point theorem it is important to be able to conclude whether or not a subset of $C(A)$ or $L^p(A)$ is compact.

Example: Let S be the set $\{f \in C([0,1]) : f(0) = 0,\ f(1) = 1, \|f\| \leq 1\}$ and the operator T defined by $T(f)(x) = f(x^2)$, $x \in [0,1]$. The norm $\|\cdot\|$ is the max-norm. It is easy to show that S is a closed bounded convex set in $C([0,1])$ and that T is a continuous mapping. Moreover it is straight-forward to show that T has no fixed point in S. The conclusion is thus that S is not a compact set in $C([0,1])$.

Our next result gives a characterization of the compact sets in $C(A)$.

Theorem 2.10 (Arzela-Ascoli theorem)**.** *Assume that K is a compact set in $\mathbb{R}^n$, $n \geq 1$ (e.g. $K = [a,b] \subset \mathbb{R}$). Then a set $S \subset C(K)$ is relatively compact in $C(K)$ iff the functions in S are uniformly bounded and equicontinuous on K.*

To say that the functions in S are uniformly bounded means that there exists a $M > 0$ such that

$$\|f\| = \sup_{x \in K} |f(x)| \leq M \quad \text{all} \;\; f \in S.$$

To say that the functions in S are equicontinuous on K means that for every $\epsilon > 0$ there exists an $\delta > 0$ such that for every $x, y \in K$ and every $f \in S$ we have

$$|x - y| < \delta \quad \Rightarrow \quad |f(x) - f(y)| < \epsilon.$$

The Arzela-Ascoli theorem can be generalized to the whole of $\mathbb{R}^n$ if we assume that the functions uniformly tends to 0 at infinity i.e. as $|x| \to \infty$.

We give the main steps in a proof of the Arzela-Ascoli theorem above.

- Show that there exits a countable dense set $\{x_1, x_2, x_3, \ldots, x_n, \ldots\}$ in K (follows from K being compact and hence has a finite ϵ-net for all $\epsilon > 0$)

- Consider an arbitrary sequence $(f_n)_{n=1}^{\infty}$ in S. From the boundedness of the sequence $(f_n)_{n=1}^{\infty}$ we can find a subsequence $(f_{n,1})_{n=1}^{\infty}$ of $(f_n)_{n=1}^{\infty}$ such that the sequence $(f_n(x_1))_{n=1}^{\infty}$ converges in $\mathbb{R}$ if the functions are real-valued. Inductively we can for a subsequence $(f_{n,k})_{n=1}^{\infty}$ of $(f_n)_{n=1}^{\infty}$ that converges at $x_1, x_2, \ldots, x_k$ find a subsequence $(f_{n,k+1})_{n=1}^{\infty}$ of $(f_{n,k})_{n=1}^{\infty}$ that also converges at x_{k+1} in $\mathbb{R}$.

- Define the subsequence $(g_n)_{n=1}^\infty$ of $(f_n)_{n=1}^\infty$ by $g_n = f_{n,n}$, $n = 1, 2, 3, \ldots$, i.e. we consider the diagonal sequence. This sequence converges at every x_k, $k = 1, 2, 3, \ldots$.
- From the equicontinuity of S we can prove that that the sequence $(g_n)_{n=1}^\infty$ converges in $S \subset C(K)$.
- These steps show the "if"-part of the Arzela-Ascoli theorem. The "only if"-part of the theorem is easy to prove.

Next we formulate a criteria for compactness for sets of L^p-functions.

Theorem 2.11 (Riesz, Kolmogorov)**.** *Assume that $1 \le p < \infty$ and that $S \subset L^p(\mathbb{R}^n)$. Then S is relatively compact in $L^p(\mathbb{R}^n)$ iff the following conditions are satisfied:*

1. *S is a bounded set in $L^p(\mathbb{R}^n)$, i.e. there exists a $M > 0$ such that $\|f\|_{L^p} \le M$ for all $f \in S$,*
2. *$\lim_{x\to 0} \int_{\mathbb{R}^n} |f(y+x) - f(y)|^p \, dy = 0$ uniformly in S, i.e. for every $\epsilon > 0$ there exists a $\delta > 0$ such that*
$$|x| < \delta \ \text{ and } \ f \in S \Rightarrow \|f(\cdot + x) - f(\cdot)\| \equiv (\int_{\mathbb{R}^n} |f(y+x) - f(y)|^p \, dy)^{1/p} < \epsilon,$$
3. *$\lim_{R\to\infty} \|f\|_{L^p(\mathbb{R}^n \setminus B(0,R))} = (\int_{|x|>R} |f(x)|^p \, dx)^{1/p} = 0$ uniformly in S, i.e. for every $\epsilon > 0$ there exists a $\omega > 0$ such that*
$$R > \omega \ \text{ och } \ f \in S \Rightarrow (\int_{|x|>R} |f(x)|^p \, dx)^{1/p} < \epsilon.$$

The above results can be found in most textbooks on functional analysis.

We are now ready to apply Schauder's theorem. Note the difference between Schauder's theorem and Banach's theorem, namely to apply Banach's theorem we have to show that a mapping is "sufficiently small", while to apply Schauder's theorem we have to prove that a mapping is compact. This means, in the $C(A)$ or L^p case, that we have to show that the image set for the mapping consists of more "regular" functions.

Example (an integral equation of Hammerstein-type): Assume that $K(x, y)$ is a continuous function for $0 \le x, y \le 1$ and that $f(y, z)$ is a bounded continuous function for $0 \le y \le 1$ and $z \in \mathbb{R}$. Then the equation
$$z(x) = \int_0^1 K(x, y) f(y, z(y)) \, dy$$
has a solution $z \in C([0, 1])$.

We want to prove that T has a fixed point where
$$(T(z))(x) = \int_0^1 K(x, y) f(y, z(y)) \, dy.$$

To show this we will apply the generalization of Schauder's fixed point theorem. We will choose a closed convex subset $S \subset C([0,1])$ such that the mapping $T : S \to C([0,1])$ is continuous and such that the image set $T(S)$ is relatively compact in $C([0,1])$.

First we observe that T maps continuous functions to continuous functions, i.e. that we have

$$T(C([0,1])) \subset C([0,1]).$$

This can be seen as follows: From the hypothesis there exists a $B > 0$ such that

$$|f(y,z)| \leq B \quad \text{if} \quad (y,z) \in [0,1] \times \mathbb{R}.$$

Moreover $K(x,y)$ is continuous on the compact set $[0,1] \times [0,1]$ and hence K is uniformly continuous on $[0,1] \times [0,1]$. Fix an $\epsilon > 0$. Then there exists a $\delta > 0$ such that

$$|K(x,y) - K(\tilde{x},\tilde{y})| < \frac{\epsilon}{B} \quad \text{if} \quad |(x,y) - (\tilde{x},\tilde{y})| < \delta.$$

Consequently for any $z \in C([0,1])$ we have

$$|(T(z))(x) - (T(z))(\tilde{x})| = |\int_0^1 (K(x,y) - K(\tilde{x},y)) f(y, z(y))\, dy| \leq$$
$$\leq \int_0^1 |K(x,y) - K(\tilde{x},y)||f(y,z(y))|\, dy \leq B \int_0^1 |K(x,y) - K(\tilde{x},y)|\, dy < \epsilon$$

provided $|x - \tilde{x}| < \delta$. This means that $T(z) \in C([0,1])$.

A natural choice for the closed convex set S is

$$S = \{z \in C([0,1]) : \|z\| \leq D\},$$

where $D > 0$ is a constant that should be chosen such that $T(S) \subset S$. We note that since K is continuous on the compact set $[0,1] \times [0,1]$ there exists an $A > 0$ such that

$$|K(x,y)| \leq A \quad \text{if} \quad (x,y) \in [0,1] \times [0,1].$$

This implies that

$$|(T(z))(x)| = |\int_0^1 K(x,y) f(y,z(y))\, dy| \leq \int_0^1 |K(x,y)||f(y,z(y))|\, dy \leq AB$$

for $z \in C([0,1])$. Hence we get

$$\|T(z)\| \leq D$$

provided we choose $D \geq AB$. Set $D = AB$. With this choice for S we get

$$T(S) \subset S.$$

To apply Schauder's theorem we have to show that $T(S)$ is relatively compact in $C([0,1])$ and that T is continuous on S. The relatively compactness is consequence of Arzela-Ascoli theorem once we have shown that $T(S)$ is uniformly bounded and equicontinuous on S.

We have above verified that $T(C([0,1]))$ is uniformly bounded and equicontinuous on S. It remains to prove that $T : S \to T(S)$ is continuous. From the definition of S it follows that

$|z(x)| \leq D$ for all $x \in [0,1]$. The continuity of $f(y,z)$ on the compact set $[0,1] \times [-D,D]$ implies that f is uniformly continuous on $[0,1] \times [-D,D]$. Fix an arbitrary $\epsilon > 0$. Then there exists a $\delta > 0$ such that

$$|f(y,z) - f(\tilde{y},\tilde{z})| < \frac{\epsilon}{A} \quad \text{if} \quad |(y,z) - (\tilde{y},\tilde{z})| < \delta.$$

Hence for arbitrary $z_1, z_2 \in S$ with $\|z_1 - z_2\| < \delta$ we have

$$\begin{aligned}
\|T(z_1) - T(z_2)\| &= \sup_{x\in[0,1]} \left| \int_0^1 K(x,y)(f(y,z_1(y)) - f(y,z_2(y)))\,dy \right| \leq \\
&\leq \sup_{x\in[0,1]} \int_0^1 |K(x,y)||(f(y,z_1(y)) - f(y,z_2(y)))|\,dy \leq \\
&\leq A \int_0^1 |(f(y,z_1(y)) - f(y,z_2(y)))|\,dy < \epsilon.
\end{aligned}$$

Now we have shown that T is continuous on S. Schauder's fixed point theorem implies that the equation $z = T(z)$ has at least one solution.

2.5 Some more fixed point theorems

We conclude this note with some additional fixed point theorems. The first one, Schaefer's fixed point theorem, is a version of Schauder's theorem. Sometimes it is called the Leray-Schauder principle and is an example of the mathematical principle saying "apriori estimates implies existence". The second one, Krasnoselskii's fixed point theorem, is a mix of Banach's and Schauder's fixed point theorems.

Theorem 2.12 (Schaefer's fixed point theorem)**.** *Assume that X is a Banach space and that $T : X \to X$ is a continuous compact*[5] *mapping. Moreover assume that the set*

$$\bigcup_{0\leq\lambda\leq 1} \{x \in X : x = \lambda T(x)\}$$

is bounded. Then T has a fixed point.

Proof. Assume that the mapping T satisfies the hypothesis in the theorem. Pick a $R > 0$ such that

$$x = \lambda T(x) \ \text{ and } \ 0 \leq \lambda \leq 1$$

implies that

$$\|x\| < R.$$

Define the mapping $\tilde{T} : X \to X$ as follows:

$$\tilde{T}(x) = \begin{cases} T(x) & \text{if } \|T(x)\| \leq R \\ \frac{R}{\|T(x)\|} T(x) & \text{if } \|T(x)\| > R \end{cases}$$

[5] T is a compact mapping if $(T(x_n))_{n=1}^{\infty}$ has a convergent subsequence for every bounded sequence $(x_n)_{n=1}^{\infty}$ in X. Usually by a compact (or completely continuous) mapping one means a continuous mapping with the property above. For linear mappings the continuity follows from this property but it is not true in general for nonlinear mappings.

This implies that $\tilde{T} : X \to X$ is a compact operator. To show this take a bounded sequence $(x_n)_{n=1}^{\infty}$ in X. Then there exists a subsequence $(x_{n_k})_{k=1}^{\infty}$ such that $\|T(x_{n_k})\| < R$ for all k or $\|T(x_{n_k})\| \geq R$ for all k. In the first case $(\tilde{T}(x_{n_k}))_{k=1}^{\infty}$ has a convergent subsequence since $\tilde{T}(x_{n_k}) = T(x_{n_k})$ and T is a compact mapping. In the second case we get that $(T(x_{n_k}))_{k=1}^{\infty}$ has a convergent subsequence, denote it by $(T(x_l))_{l=1}^{\infty}$ for convenience. But then it follows that also $(\|T(x_l)\|)_{l=1}^{\infty}$ converges, where also $\|T(x_l)\| \geq R$ for all l. Hence we have $\tilde{T}(x_l) = \frac{R}{\|T(x_l)\|} T(x_l)$.

Set

$$K = \overline{\operatorname{co} \overline{\tilde{T}(B(0,R))}}.$$

Here K is convex (it is the closed convex hull of a set), compact (the convex hull of a compact set is compact and $\tilde{T}$ is a compact mapping) subset of X such that

$$\tilde{T} : K \to K.$$

Schauder's fixed point theorem implies that $\tilde{T}$ has a fixed point $x_0 \in K$. But x_0 is a fixed point for T if $\|T(x_0)\| \leq R$. Assume that $\|T(x_0)\| > R$. This yields a contradiction since $x_0 = \tilde{T}(x_0) = \lambda T(x_0)$, where $\lambda = \frac{R}{\|T(x_0)\|} \in (0,1)$, since according to the hypothesis of the theorem it should follow that $\|T(x_0)\| = \|x_0\| < R$. This proves the theorem. □

Note that to apply Schaefer's theorem we do not need to prove that a certain set is convex or compact. The problem is reformulated as to show a certain a priori estimate for the operator T.

Theorem 2.13 (Krasnoselskii's fixed point theorem)**.** *Assume that F is a closed bounded convex subset of a Banach space X. Furthermore assume that T_1 and T_2 are mappings from F into X such that*

1. *$T_1(x) + T_2(y) \in F$ for all $x, y \in F$,*
2. *T_1 is a contraction,*
3. *T_2 is continuous and compact.*

Then $T_1 + T_2$ has a fixed point in F.

Proof. Assume that the mappings T_1, T_2 satisfies the hypothesis of the theorem. In particular there exists a $c \in (0,1)$ such that

$$\|T_1(x) - T_1(y)\| \leq c\|x - y\|, \quad x, y \in F.$$

This yields

$$\|(I - T_1)(x) - (I - T_1)(z)\| \geq \|x - z\| - \|T_1(x) - T_1(z)\| \geq (1 - c)\|x - z\|$$

and

$$\|(I - T_1)(x) - (I - T_1)(z)\| \leq \|x - z\| + \|T_1(x) - T_1(z)\| \leq (1 + c)\|x - z\|.$$

Consequently $I - T_1 : F \to (I - T_1)(F)$ is a homeomorphism, and $(I - T_1)^{-1}$ exists as a continuous mapping from $(I - T_1)(F)$. Furthermore we note that for each $y \in F$ the equation

$$x = T_1(x) + T_2(y)$$

has a unique solution $x \in F$ according to Banach's fixed point theorem. From this we conclude that $T_2(y) \in (I - T_1)(F)$ for every $y \in F$ and also that $(I - T_1)^{-1}T_2 : F \to F$ is a well-defined continuous mapping. Since T_2 is a compact mapping it follows that $(I - T_1)^{-1}T_2 : F \to F$ is a compact mapping. Finally the generalization of Schauder's fixed point theorem yields the conclusion of the theorem. $\square$

We recommend anyone interested in fixed point theorems to browse through the books [2] and [1] where additional results and many more references can be found.

References

[1] A.Granas, *Fixed point theory*, Springer 2003

[2] D.R.Smart, *Fixed Point Theorems*, Cambridge Univ. Press 1973

3 L^p-spaces

3.1 Introduction

A basic feature for the important results in this course – Banach's fixed point theorem, Brouwer's fixed point theorem, Schauder's fixed point theorem, Hilbert-Schmidt theorem and others – is that the mappings that appear should be defined on **complete** normed spaces = Banach spaces. The completeness is crucial and the theorems would no longer be true without the assumption on completeness.

A technique often used to prove existence of a solution to a problem (and also to find the solution) is to find solutions to approximate problems and by improving the approximations it can sometimes be possible to obtain a sequence of approximative solutions that forms a Cauchy sequence in a proper space. A solution to the original problem can then often be obtained as the limit of the Cauchy sequence provided the space is a Banach space.

An example of a function space that often appears is the vector space of all continuous functions defined on $\mathbb{R}^n$ or some "nice"[6] subset Ω of $\mathbb{R}^n$, with pointwise defined addition and multiplication by scalars. We note the if $C(\Omega)$ is equipped with the sup-norm, i.e.

$$\|f\| = \sup_{t\in\Omega} |f(t)|, \quad f \in C(\Omega),$$

then the normed space $(C(\Omega), \|\cdot\|)$ becomes a Banach space. But if $C(\Omega)$ is supplied with the norm

$$\|f\|_1 = \int_\Omega |f(t)|\, dt, \quad f \in C(\Omega),$$

then $(C(\Omega), \|\cdot\|_1)$ is a normed space but **not** a Banach space. See for instance example 1 below. The set Ω is supposed to be a compact subset of $\mathbb{R}^n$ so all integrals are finite. It is a pity that $(C(\Omega), \|\cdot\|_1)$ is not a Banach space since the norm $\|\cdot\|_1$ gives a natural measure of size. If f is a density function then $\|f\|_1$ corresponds to the total mass. Moreover in physics the integral

$$\|f\|_2 = (\int_\Omega |f(t)|^2\, dt)^{1/2}, \quad f \in C(\Omega).$$

measures the "energy" of a system described by f. In general it is natural to consider norms

$$\|f\|_p = (\int_\Omega |f(t)|^p\, dt)^{1/p}, \quad f \in C(\Omega),$$

where $p \in [1, \infty)$. To see that these expressions really define norm functions the same technique that was used to prove the corresponding statements for the sequence spaces l^p can be used.

Example 1: Consider the set $\Omega = [0, 1] \subset \mathbb{R}$ and define

$$f_n(t) = \begin{cases} 0 & t \in [0, \frac{1}{2}) \\ 2n(t - \frac{1}{2}) & t \in [\frac{1}{2}, \frac{1}{2} + \frac{1}{2n}) \\ 1 & t \in [\frac{1}{2} + \frac{1}{2n}, 1] \end{cases}$$

[6] We assume that Ω is compact and equal to the closure of its interior.

for $n = 1, 2, \ldots$ Sketch the graph for f_n here!

We see that $(f_n)_{n=1}^{\infty}$ defines a Cauchy sequence in the normed space $(C[0,1], \| \ \|_1)$ since

$$\|f_n - f_m\|_1 = \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{2\min(n,m)}} |f_n(t) - f_m(t)|\, dt$$

$$\leq \frac{1}{2\min(n,m)} \to 0, \quad n, m \to \infty.$$

However there is no continuous function f such that $f_n \to f$ in $(C[0,1], \| \ \|_1)$. Prove this! On the other hand the sequence $(f_n)_{n=1}^{\infty}$ converges pointwise to h given by

$$h(t) = \begin{cases} 0 & t \in [0, \frac{1}{2}] \\ 1 & t \in (\frac{1}{2}, 1] \end{cases}.$$

h is not continuous but still Riemann integrable and satisfies

$$\lim_{n\to\infty} \|f_n - h\|_1 = 0.$$

The fact that the function h above is Riemann integrable might suggest that

$$(\text{Riemann integrable functions}, \| \cdot \|_1) \tag{19}$$

is a Banach space. It is clear that linear combinations of Riemann integrable functions are Riemann integrable and that also products of Riemann integrable functions are Riemann integrable[7]. However Riemann integrable functions are not closed under pointwise limits as seen from the following example.

Example 2: Let Ω denote the interval $[0,1]$ and let $\{r_1, r_2, r_3, \ldots\}$ be an enumeration of all rational numbers in the interval $[0,1]$. For $n = 1, 2, \ldots$ define

$$f_n(t) = \chi_{\{r_1,\ldots,r_n\}}(t) = \begin{cases} 1 & t \in \{r_1, \ldots, r_n\} \\ 0 & t \notin \{r_1, \ldots, r_n\} \end{cases}.$$

Moreover set

$$f(t) = \chi_{\{r_1,r_2,r_3,\ldots\}}(t) = \begin{cases} 1 & t \in \{r_1, r_2, r_3, \ldots\} \\ 0 & t \notin \{r_1, r_2, r_3, \ldots\} \end{cases}.$$

[7] If f is Riemann integrable then so is f^2 and if both f and g are Riemann integrable then so is fg since $fg = \frac{1}{4}((f+g)^2 - (f-g)^2)$.

We note that f_n is Riemann integrable for every n and that $(f_n)_{n=1}^{\infty}$ is a Cauchy sequence in the normed space (Riemann integrable functions, $\|\cdot\|_1$), but the pointwise limit function f is not Riemann integrable. Prove this! Here should also be observed that $\|\cdot\|_1$ cannot see the difference between f_n for any n and $\mathbb{0}$. Here $\mathbb{0}$ denotes the function that is pointwise 0 for all $x \in [0,1]$. For $\|\cdot\|_1$ to be a norm for the Riemann integrable functions we have to **identify** f_n and $\mathbb{0}$ for all n.

So if we want to have a Banach space containing all Riemann integrable functions we ought accept f as an element in that space since it is the pointwise limit of the sequence $(f_n(x))$ (we have $0 \le f_n(x) \uparrow f(x) \le 1$ for all $x \in [0,1]$). In applications it will be important for us to have strong convergence theorems of the form

$$" \lim_{n\to\infty} \int f_n \, dx = \int \lim_{n\to\infty} f_n \, dx ".$$

If this holds for the functions considered above we note that $f \neq \mathbb{0}$ but neither the less we have $\|f - \mathbb{0}\|_1 = 0$. We can not detect the difference between f and $\mathbb{0}$ measuring with the $\|\cdot\|_1$–norm and have to **identify** these functions. We will say say that the functions differs on a set of measure 0. This identification also has to be done for Riemann integrable functions for $\|\cdot\|_1$ to be a norm.

Considering the $\|\cdot\|_p$–norms, $1 \le p < \infty$ in general, it can be observed that only for $p = 2$ the norm is a Hilbert space norm, i.e. there can be defined an inner product $\langle \cdot, \cdot \rangle$ on the vector space in such a way that $\|x\| = \sqrt{\langle x, x \rangle}$ for all x holds true. Neither the sup-norm can be connected with an inner product. The Hilbert space structure will be important to us in connection with spectral theory in chapter 4 in [1].

The problem is now to extend the normed space $(C(\Omega), \|\cdot\|_p)$ to a Banach space. The method to complete the normed space that is discussed in section 4 chapter 1 in [1] has the disadvantage that the properties of the elements in the completion can be hard to read off and it is not obvious that the elements are pointwise defined functions.

However let us quickly remind ourself of the constuction for a completion of a normed space that is given in [1]: Given a normed space $(E, \|\cdot\|)$ let $\tilde{E}$ be the set of all equivalence classes of Cauchy sequences in E, denoted $[(x_n)]_\sim$, where

$$(x_n)_{n=1}^{\infty} \sim (y_n)_{n=1}^{\infty} \text{ if } \lim_{n\to\infty} \|x_n - y_n\| = 0.$$

We will below make a quick gallop through the landscape of Lebegue integration with stops at measurable sets, measurable functions, Lebesgue integrals, convergence theorems and L^p–spaces. Some ideas for the proofs will be sketched. For those who are interested in a thorough treatment we refer to the books by Folland [3] (textbook on graduate level), Rudin [5] (also a graduate level textbook), Rudin [5] (a more elementary book), Apostol [1] (has been used for undergraduate courses at GU) or why not Hörmander [4]. The presentations differ slightly but most are based on measure theory.

3.2 Lebesgue measure on $\mathbb{R}^n$

In measure theory we want to generalize the concept of length of an interval in $\mathbb{R}$, area of a rectangle in $\mathbb{R}^2$ on so on to a wider class of sets. The ultimate goal is to assign a measure to as many sets as possible where the measure has to satisfy certain natural conditions. If all intervals $[a, b] \subset \mathbb{R}$, $a < b$ (as well as the intervals $[a, b)$, $(a, b]$, (a, b)) should have the measure $b - a$ then there are subsets of the real numbers that are impossible to assign a measure to[8]. This is hard to prove and is based on the axiom of choice[9].

First let us see for which subsets of a an arbitrary set X it would be natural to be able to assign measure to. Intuitively it is natural that given countable many sets, where all have a well-defined measure, all sets that can be obtained by countably many applications with the set operations union, intersection and complement should also be possible to assign a measure to. This motivates the following definition.

Definition 3.1. *A set $\mathcal{M}$ of subsets of X is called a σ–algebra if*

1. $\emptyset \in \mathcal{M}$
2. $E \in \mathcal{M}$ *implies* $X \setminus E \in \mathcal{M}$
3. $E_1, E_2, \ldots \in \mathcal{M}$ *implies* $\bigcup_{n=1}^{\infty} E_n \in \mathcal{M}$

A set in $\mathcal{M}$ is called a measurable set. Let $\mathcal{B}_n$ denote the smallest σ–algebra that contains all open sets in $\mathbb{R}^n$. This is called the **Borel σ–algebra**. For simplicity we restrict to the case $n = 1$ but what is said holds true for general n. There exists such a smallest σ–algebra, since the intersection of any collection of σ–algebras is a σ–algebra, and all the intervals of the four types above are contained here.

Given a σ–algebra $\mathcal{M}$ we can talk about a measure μ on $\mathcal{M}$. A measure should satisfy some properties encoded in the next definition.

Definition 3.2. *A measure μ on the σ–algebra $\mathcal{M}$ is a mapping*

$$\mu : \mathcal{M} \to [0, +\infty]$$

such that

1. $\mu(\emptyset) = 0$
2. $E_1, E_2, \ldots \in \mathcal{M}$ *mutually disjoint sets implies* $\mu(\bigcup_{n=1}^{\infty} E_n) = \Sigma_{n=1}^{\infty} \mu(E_n)$

[8] A well-known example of this is due to Vitali. Even more striking is the following example in $\mathbb{R}^3$ by Banach and Tarski and which only involves finite additivity. They proved: The unit ball in $\mathbb{R}^3$ can be decomposed into a finite number of pieces which may be reassembled, using only translation and rotation, to form 2 disjoint copies of the unit ball

[9] The axiom of choice says that for every class of non-empty sets E_λ, $\lambda \in \Lambda$, there exists a set consisting of one element from every set E_λ.

The property 2. is called countable additivity for measures and is a key property when defining Lebesgue integrals. The main question is now whether it is possible to prove the existence of a unique measure on the Borel σ–algebra with the property that all intervals with end points at a and b has the measure $|b-a|$. The answer is yes and this measure is called the **Borel measure**. This is the foundation on which the L^p-theory rests. The **Lebesgue measure** is obtained by completing the Borel measure in the following sense.

Definition 3.3. *Let μ be a measure on a σ–algebra $\mathcal{M}$. Then there exists a σ–algebra $\bar{\mathcal{M}}$ and a well-defined measure $\bar{\mu} : \bar{\mathcal{M}} \to [0, +\infty]$ such that $E \in \bar{\mathcal{M}}$ iff $E = A \bigcup B$, where $A \in \mathcal{M}$ and $B \subset C \in \mathcal{M}$ with $\mu(C) = 0$, and $\bar{\mu}(E) = \mu(A)$.*

What has been done is to add all subsets of measurable sets with measure 0 in such a way that also $\bar{\mathcal{M}}$ becomes a σ–algebra. Note that it follows from the definition that if $A, B \in \mathcal{M}$, $A \subset B$, then we have $\mu(A) \leq \mu(B)$. We call $\bar{\mathcal{B}}_1$ the **Lebesgue σ–algebra** on $\mathbb{R}$ and denote it by $\mathcal{L}_1$ and the completed Borel measure on $\mathcal{L}_1$ denoted m is called the **Lebesgue measure**.

We mentioned above that there are subsets of $\mathbb{R}$ that are not Lebesgue measurable. The following result can be proved.

Theorem 3.1 (Approximation)**.** *Let $E \subset \mathbb{R}$ be Lebesgue measurable. Then we have*

$$m(E) = \inf\{m(U) : E \subset U,\ U \text{ open}\} = \sup\{m(K) : K \subset E,\ K \text{ compact}\}.$$

Moreover if $m(E) < \infty$ then for every $\epsilon > 0$ there exists an open set A consisting of finitely many open intervals such that

$$m((E \setminus A) \bigcup (A \setminus E)) < \epsilon.$$

What has been said about $\mathbb{R}$ is true for $\mathbb{R}^n$, $n = 2, 3, \ldots$, provided intervals are replaced by rectangles parallel to the axis etc. By $\mathcal{L}_n = \mathcal{L}$ we denote the Lebesgue σ–algebra on $\mathbb{R}^n$, i.e. the completed Borel σ–algebra $\mathcal{B}_n$, and the approximation theorem above corresponds to a natural generalization for $\mathbb{R}^n$.

3.3 Lebesgue measurable functions

We will now consider functions f that takes values in $\bar{\mathbb{R}} = \mathbb{R} \bigcup \{\pm\infty\}$ where we define $0 \cdot \infty = 0$. What has to be avoided is undefined expressions like $\infty - \infty$. In this section every function takes values in $\bar{\mathbb{R}}$. We say that the function $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ is **Lebesgue measurable** if $f^{-1}([a, \infty)) \in \mathcal{L}$ for every $a \in \mathbb{R}$. Here $f^{-1}(U)$ denotes the set $\{x \in \mathbb{R}^n : f(x) \in U\}$, i.e. the inverse image of U under f. From this definition it follows that $f^{-1}(E) \in \mathcal{L}$ for every Borel set E but also that all functions that can be formed using the operations

$$+ \quad \cdot \quad \sup_{n=1,2,\ldots} \quad \inf_{n=1,2,\ldots} \quad \limsup_{n=1,2,\ldots} \quad \liminf_{n=1,2,\ldots}$$

on Lebesgue measurable functions are Lebesgue measurable. More precisely, given Lebesgue measurable functions f, g, f_n, $n = 1, 2, \ldots$ then the functions

1. $f+g$, fg, λf, where $\lambda \in \mathbb{R}$
2. $\max(f,g)$, $\min(f,g)$
3. $\sup_{n=1,2,\ldots} f_n$, $\inf_{n=1,2,\ldots} f_n$
4. $\limsup_{n\to\infty} f_n \equiv \lim_{k\to\infty} \sup_{n\geq k} f_n$, $\liminf_{n\to\infty} f_n \equiv \lim_{k\to\infty} \inf_{n\geq k} f_n$

are also Lebesgue measurable. It can be shown that every continuous function is Lebesgue measurable! The most important examples of Lebesgue measurable functions are the so called **simple functions** that are given by finite linear combinations of characteristic functions for Lebesgue measurable sets, i.e. functions of the form

$$\Sigma_{n=1}^{N} \lambda_n \chi_{E_n}$$

where $\chi_E(t) = 1$ if $t \in E$ and $= 0$ if $t \notin E$. We assume that $\lambda_i \neq \lambda_j$ for $i \neq j$. Check for yourself that the simple functions are Lebesgue measurable. The key property for the simple functions is the following observation.

Theorem 3.2 (Approximation)**.** *Let $f : \mathbb{R}^n \to [0,\infty]$ be a Lebesgue measurable function. Then there exists a sequence of simple functions ϕ_n, $n = 1, 2, \ldots$ such that*

1. $0 \leq \phi_1 \leq \phi_2 \leq \phi_3 \leq \ldots$

2. $\lim_{n\to\infty} \phi_n(t) = f(t)$ *for all* $t \in \mathbb{R}^n$

3. ϕ_n *converges uniformly to* f *on each set* $A \subset \mathbb{R}^n$ *where* f *is bounded.*

We note that the limit function for an increasing sequence of simple functions is also Lebesgue measurable. But also converse, i.e. that every Lebesgue measurable function (bounded below) can be obtained as the limit function for an increasing sequence of simple functions.

The proof for the theorem is quite simple. Set

$$\phi_n = \Sigma_{k=0}^{2^{2n-1}} k2^{-n}\chi_{E_n^k} + 2^n \chi_{F_n},$$

where

$$E_n^k = f^{-1}((k2^{-n}, (k+1)2^{-n}])$$

and

$$F_n = f^{-1}((2^n, \infty]).$$

for $n = 1, 2, \ldots$. For an f of your choice draw the graphs for ϕ_n!

Next we introduce the term **almost everywhere**, abbreviated *a.e.*, which means everywhere except on a set of measure 0. To say that the functions f and g are equal *a.e.* means that the set where the functions differ must not be empty but have the Lebesgue measure 0. In the same way $f_n \to f$ pointwise *a.e.* means that the set where we do not have convergence is a 0–set. Since every subset of a 0–set is a 0–set we get

1. f Lebesgue measurable and $f = g$ *a.e.* implies that g is Lebesgue measurable.

2. f_n, $n = 1, 2, \ldots$, Lebesgue measurable and $f_n \to f$ pointwise *a.e.* implies that f is Lebesgue measurable.

Finally $f : \mathbb{R}^n \to \mathbb{C}$ is called Lebesgue measurable if both $\mathrm{Re} f$ and $\mathrm{Im} f$ are Lebesgue measurable. This is the same as saying that $f^{-1}(U) \in \mathcal{L}$ for every open set U in $\mathbb{C}$.

3.4 Integrals and convergence theorems

A complex-valued function f can uniquely be written as a sum of its real- and imaginary part

$$f = \mathrm{Re} f + i\, \mathrm{Im} f,$$

where $\mathrm{Re} f$ and $\mathrm{Im} f$ are real-valued. Both these functions can be written as a sum of the positive and the negative part of f. If f is real-valued we denote

$$f^+ = \max(f, 0)$$

and

$$f^- = \max(-f, 0).$$

Hence we get $f = f^+ - f^-$ (and $|f| = f^+ + f^-$). Since we want the integral operator

$$f \mapsto \int f\, dm$$

(not yet defined) to be linear on Lebesgue integrable functions we must have

$$\int f\, dm = \int (\mathrm{Re} f)^+\, dm - \int (\mathrm{Re} f)^-\, dm + i(\int (\mathrm{Im} f)^+\, dm - \int (\mathrm{Re} f)^-\, dm).$$

So it is enough to define

$$\int f\, dm$$

for all Lebesgue measurable functions $f : \mathbb{R}^n \to [0, \infty]$. This will be done in two steps.

Step 1 For $f = \Sigma_{n=1}^N \lambda_n \chi_{E_n}$, i.e. for a simple function f, we set

$$\int f\, dm = \Sigma_{n=1}^N \lambda_n m(E_n).$$

Step 2 If f is a Lebesgue measurable function we set

$$\int f\, dm = \sup\{\int \phi\, dm : \phi \text{ simple function, } 0 \leq \phi \leq f\}.$$

It can quite easily be shown that the integral is well-defined. The integral can attain the value $+\infty$ since we have not assumed any size condition for f. We let L^+ denote the set of all real Lebesgue measurable functions that takes values in $[0, \infty]$.

From the definition it follows that $f, g \in L^+$ and $f \leq g$ implies

$$\int f\, dm \leq \int g\, dm.$$

Moreover we let L^1 denote the set of all Lebesgue measurable functions $f : \mathbb{R}^n \to \mathbb{C}$ for which

$$\max(\int \mathrm{Re} f^+\, dm, \int \mathrm{Re} f^-\, dm, \int \mathrm{Im} f^+\, dm, \int \mathrm{Im} f^-\, dm) < \infty.$$

This is equivalent to

$$\int |f|\, dm < \infty.$$

Moreover we note that

$$|\int f\, dm| \leq \int |f|\, dm.$$

Finally we define

$$\int_E f\, dm = \int f \chi_E\, dm,$$

for E a Lebesgue measurable set in $\mathbb{R}^n$.

The question is then: What is the difference between the **Lebesgue integral** and the **Riemann integral**? The answer sits in the definitions. Let us for the moment assume that f attains its values in $[0, M]$ for some $M > 0$. Remember that the definition of the Riemann integral is based on splitting the x–axis into a union of tiny disjoint intervals I_k. Set $M_k = \sup_{I_k} f$ and $m_k = \inf_{I_k} f$. We get

$$\int f\, dx \approx \Sigma_k M_k |I_k|$$

provided $\Sigma_k M_k |I_k| \approx \Sigma_k m_k |I_k|$, where $|I_k|$ denotes the length of the interval I_k. With the notation $\approx$ we mean that the difference tends to 0 as $\sup_k |I_k| \to 0$. To have this we need f to be almost constant on every interval I_k (i.e. $M_k - m_k \approx 0$) or that the number of all intervals for which this is not true (i.e. $M_k - m_k \not\approx 0$) is small. Another way to phrase it is that f should be continuous except for a small set of points with Lebesgue measure 0. Using our special lingo we say that f is Riemann integrable if the set where f is discontinuous is a set of Lebesgue measure 0. In the previous example with $f = \chi_{\mathbb{Q} \bigcap [0,1]}$ the set of points of discontinuity is the whole interval $[0, 1]$ which has Lebesgue measure equal to 1 and not 0.

The definition of the Lebesgue integral is based on splitting the y–axis into small intervals $I_n^k = [k2^{-n}, (k+1)2^{-n})$. Here n indicates how fine the decomposition is, more precisely 2^{-n} is the length of the intervals. Comparing with the definition for simple functions (we assume that f is non-negative and $M < 2^n$) we have

$$\int f\, dm \approx \Sigma_k k 2^{-n} m(E_n^k),$$

where we observe that $|f - \phi_n| \leq 2^{-n}$ on E_n^k. See page 6. For the sum to have a meaning it is needed that $m(E_n^k)$ and $m(F_n)$ are well-defined, which is guaranteed by the assumption that f is Lebesgue measurable. If we return to the function $f = \chi_{\mathbb{Q} \bigcap [0,1]}$ we see that f is 0 except at the rational points in the interval $[0, 1]$, which is equal to $\{r_1, r_2, \ldots\}$. But every set $\{r_n\}$ is Lebesgue measurable with the measure 0 and a countable union of 0–sets is a 0–set. Hence we have $\int f\, dm = 0$.

We observe that every continuous function which is different from 0 only in a compact subset of $\mathbb{R}^n$ is Riemann integrable, that each Riemann integrable function (with finite $\|\cdot\|_1$–norm) is Lebesgue integrable **and**

$$\int_{\mathbb{R}^n} f(x)\, dx = \int f\, dm.$$

Here the LHS denotes the Riemann integral for f and the RHS denotes the Lebesgue integral for f.

Below we list some theorems that will become important to us for applications. It is important to note that the Lebesgue integral is an extension for the Riemann integral with the properties we wanted: powerful convergence theorems and the function space $(L^1, \|\cdot\|_1)$ is complete.

Theorem 3.3 (Lebesgue's monotone convergence theorem). *Let $(f_n)_{n=1}^{\infty} \subset L^+$ be a monotone increasing sequence of functions. Then we have*

$$\lim_{n\to\infty} \int f_n\, dm = \int \lim_{n\to\infty} f_n\, dm.$$

Theorem 3.4 (Fatou's lemma). *Let $(f_n)_{n=1}^{\infty} \subset L^+$ be a sequence of functions. Then we have*

$$\int \liminf_{n\to\infty} f_n\, dm \le \liminf_{n\to\infty} \int f_n\, dm.$$

Theorem 3.5 (Lebesgue's dominated convergence theorem). *Assume that $(f_n)_{n=1}^{\infty}$ is a sequence of complex-valued Lebesgue measurable functions such that $\lim_{n\to\infty} f_n = f$ a.e. Moreover assume that there exists a Lebesgue measurable function g such that*

$$|f_n| \le g \in L^1 \qquad \text{all } n.$$

Then we have

$$f \in L^1$$

and

$$\lim_{n\to\infty} \int f_n\, dm = \int f\, dm.$$

Theorem 3.6 (Differentiation under the integral sign). *Assume that $f(t, x) : \mathbb{R}^n \times [a, b] \to \mathbb{C}$ and that $f(\cdot, x) : \mathbb{R}^n \to \mathbb{C}$ is a L^1–function for each $x \in [a, b]$. Set $F(x) = \int f(t, x)\, dm(t)$.*

- *Assume that there exists a $g \in L^1$ such that*

  $$|f(t, x)| \le g(t) \quad \text{all } t, x.$$

  Then we have

  $$\lim_{x\to x_0} F(x) = F(x_0)$$

  provided

  $$\lim_{x\to x_0} f(t, x) = f(t, x_0) \quad \text{all } t.$$

- *Assume that* $\frac{\partial f}{\partial x}$ *exists and that there is a* $g \in L^1$ *such that*

$$|\frac{\partial f}{\partial x}(t,x)| \leq g(t) \quad \text{all } t, x.$$

Then F *is differentiable and*

$$F'(x) = \int \frac{\partial f}{\partial x}(t,x)\, dm(t).$$

We now assume that $n = 1$ and recall that a real function is continuously differentiable iff

$$f(x) = \int_a^x g(t)\, dt$$

where g is a continuous real function. Furthermore we have $f' = g$. What can be said about the function

$$\int_a^x g(t)\, dt$$

where $g \in L^1$?

To answer this question we introduce the concept of **absolutely continuous function**. We say that the real function f is absolutely continuous if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\Sigma |b_n - a_n| < \delta$$

implies that

$$\Sigma |f(b_n) - f(a_n)| < \epsilon.$$

In particular this means that f is continuous and moreover uniformly continuous on the set where it is defined. $\Sigma \ldots$ stands for the sum for a finite series.

Theorem 3.7. *A real function* $f(x)$ *is given by* $\int_a^x g\, dm$*, where* g *is a locally*[10] *Lebesgue integrable function, iff* f *is absolutely continuous. In that case we have* $f' = g$ *a.e..*

From calculus course we remember that multiple Riemann integrals can be calculated by repeated integration. Is this still true for multiple Lebesgue integrals? The answer is contained in the following result.

Theorem 3.8 (Fubini–Tonelli's theorem)**.** *Assume that* $f(\cdot,\cdot)$ *is Lebesgue measurable and that one of the following conditions are satisfied:*

1. *(Tonelli)* $f \geq 0$
2. *(Fubini) one of the integrals* $\int |f(x,y)|\, dm(x,y)$, $\int (\int |f(x,y)|\, dm(y))\, dm(x)$, $\int (\int |f(x,y)|\, dm(x))\, dm(y)$ *is finite.*

Then the functions $f(\cdot,y)$, $f(x,\cdot)$, $\int f(\cdot,y)\, dm(y)$ *and* $\int f(x,\cdot)\, dm(x)$ *are Lebesgue measurable and*

$$\int f(x,y)\, dm(x,y) = \int (\int f(x,y)\, dm(y))\, dm(x) = \int (\int f(x,y)\, dm(x))\, dm(y).$$

[10] L^1_{loc} is defined below.

3.5 L^p–spaces, Hölder's and Young's inequalities

For Lebesgue measurable functions f we define

$$\|f\|_p = (\int |f|^p \, dm)^{1/p}, \quad p \in [1, \infty)$$

and

$$\|f\|_\infty = \operatorname{ess\,sup} |f|.$$

Here ess sup for real-valued non-negative functions f denotes the quantity

$$\operatorname{ess\,sup} f = \inf\{k : k \geq f \ \text{ a.e.}\}.$$

We now define the L^p–space as the set of all Lebesgue measurable functions such that $\|f\|_p < \infty$. This is valid for $1 \leq p \leq \infty$ and we see that

- $\|f - g\|_p = 0$ iff $f = g$ *a.e.* Functions in L^p are identified if they are equal *a.e.*
- $f \in L^p$ implies that $|f| < \infty$ *a.e.*
- $f \in L^\infty$ and f continuous implies that $\|f\|_\infty = \sup |f|$. If f is not continuous then we obtain that the set of all x where $f(x) > \|f\|_\infty$ is a 0–set.

We claim that $\|\cdot\|_p$ really defines a norm. If $p = 1, \infty$ this is trivial. For $p \in (1, \infty)$ it is a consequence of **Hölder's inequality**

$$\|fg\|_1 \leq \|f\|_p \, \|g\|_q,$$

where $\dfrac{1}{p} + \dfrac{1}{q} = 1$, $p > 1$. This is established with a similar technique to that which was used for proving the corresponding statement for the sequence space l^p. This yields the **Minkowski's inequality**

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

for $p \in (1, \infty)$.

We have now defined L^p as a normed space. The notations L^p_{loc} denotes the set of all Lebesgue measurable functions f for which $f\chi_E \in L^p$ for all compact Lebesgue measurable sets E in $\mathbb{R}^n$.

Theorem 3.9. *L^p with the norm $\|\cdot\|_p$ is a Banach space for $p \in [1, \infty]$. It is separable (there exists a countable dense set) for $p \in [1, \infty)$. If $(f_n)_{n=1}^\infty$ is a Cauchy sequence in L^p for $p \in [1, \infty)$ there exists a subsequence $(f_{n_k})_{k=1}^\infty$ that converges pointwise a.e.*

Try to prove this!!

Let f be a complex-valued function on $\mathbb{R}^n$. The closure of the set $\{x : f(x) \neq 0\}$ is called the **support** for f and we let C_0^∞ denote the set of all infinitely continuously differentiable functions with compact support.

Theorem 3.10. *For $p \in [1, \infty)$ we have*

1. $L^p \bigcap \{$ *simple functions* $\}$
2. C_0^∞

are both dense in L^p.

Finally we give some inequalities that can come in handy in many calculations.

Theorem 3.11 (Young's inequality)**.** *Assume that* $k : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ *is Lebesgue measurable and that*

$$\max(\sup_x \int |k(x,y)|\, dm(y), \sup_y \int |k(x,y)|\, dm(x)) = M < \infty.$$

If $f \in L^p$ *for some* $p \in [1, \infty]$ *then*

$$F(x) = \int k(x,y) f(y)\, dm(y)$$

belongs to L^p *and*

$$\|F\|_p \leq M \|f\|_p.$$

Theorem 3.12 (Chebyshev's inequality)**.** *Let* $f \in L^p$, $p \in [1, \infty)$ *and* $\alpha > 0$ *be given. Then we have*

$$m(\{x : |f(x)| > \alpha\}) \leq (\frac{\|f\|_p}{\alpha})^p.$$

In hindsight we note that functions that are Lebesgue integrable can be very wild but at the same time there are continuous nice functions that are close to the wild beasts in L^p-norm. Often in applications we want to prove that a certain function, appearing as a solution to some say integral equation, is continuous but from the first consideration we just obtain it as an element in L^p. However the continuity property for the function can then be established from the specific problem. What the L^p–theory has contributed with is the existence of a function that can be proven to have some good properties.

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4 Spectral theory

4.1 Introduction

In this course we focus on equations of the form

$$Af = g, \tag{20}$$

where A is a linear mapping, typically an integral operator, and g is a given element in some normed space which "almost everywhere" in text is a Banach space. The type of questions one usually and naturally poses are:

1. Is there a solution to $Af = g$ and, if so is it unique?
2. If the RHS g is slightly perturbed, i.e. data $\tilde{g}$ is chosen close to g, will the solution $\tilde{f}$ to $A\tilde{f} = \tilde{g}$ be close to f?
3. If the operator $\tilde{A}$ is a good approximation of the operator A will the solution $\tilde{f}$ to $\tilde{A}\tilde{f} = g$ be a good approximation for f to $Af = g$?

These questions will be made more precise and to some extent answered below.

One direct way to proceed is to try to calculate the "inverse operator" A^{-1} and obtain f from the expression

$$f = A^{-1}g.$$

Earlier we have seen examples of this for the case where A is a small perturbation of the identity mapping on a Banach space X, more precisely for $A = I + B$ with $\|B\| < 1$. Here the inverse operator A^{-1} is given by the Neumann series

$$\Sigma_{n=0}^{\infty}(-1)^n B^n,$$

(B^0 should be interpreted as I). We showed that if $B \in \mathcal{B}(X, X)$, where X is a Banach space, then we got $C \equiv \Sigma_{n=0}^{\infty}(-1)^n B^n \in \mathcal{B}(X, X)$ and

- $(I + B)C = C(I + B) = I_X$
- $\|C\| \leq \frac{1}{1-\|B\|}$

Sometimes we are able to prove the existence and uniqueness for solutions to equations $Af = g$ in Banach spaces X without being able to explicitly calculate the solution. An example is as follows: *Assume there exists a family $\{A_t\}_{t\in[0,1]}$ of bounded linear operators on a Banach space that satisfies the following conditions: There exists a positive constant C such that*

1. *$\|f\| \leq C\|A_t f\|$ for all $t \in [0, 1]$ and all $f \in X$*
2. *$\|A_t f - A_s f\| \leq C|t - s|\,\|f\|$ for all $t, s \in [0, 1]$ and all $f \in X$*
3. *A_0 is an invertible operator on X, where the inverse is a bounded linear operator on X.*

Then we can conclude that also A_1 is an invertible operator on X, where the inverse is a bounded linear operator on X. This is a general method and is referred to as the **method of continuity**. The idea here is that A_1 is a perturbation of the "nice" invertible operator A_0, where the perturbation is controlled by conditions 1 and 2 above. The proof of the statement, essentially that A_1 is surjective, is given below.

Before we proceed let us make a clarification concerning the use of different notions.

When we talk about an operator A we do not a priori assume that A is linear (even if it is a mapping between two vector spaces) though it is in most applications.

What do we mean by an "inverse operator"? If we consider A as a mapping $X \to Y$ it is enough for A to be injective, i.e. $A(x) = A(y)$ implies $x = y$, for A^{-1} to be defined as a mapping $\mathcal{R}(A) \to X$. Here $\mathcal{R}(A) = \{y \in Y : y = A(x) \text{ for some } x \in X\}$ is a subset of Y. The injectivity implies that the equation (21) has at most one solution, viz. if $g \in \mathcal{R}(A)$ there exists a unique solution otherwise there is no solution. Moreover if A is surjective, i.e. $\mathcal{R}(A) = Y$, then the equation has a unique solution f for every $g \in Y$. So if we consider A in the category of mappings we say that A^{-1}, called the **inverse mapping** to A, exists if the equation (21) has a unique solution for every RHS, i.e. $A^{-1}(f) = g$.

However if X and Y are normed spaces and A is a bounded linear mapping we could look for a mapping B such that

$$AB = I_{\mathcal{R}(A)}, \quad BA = I_X$$

with the additional properties to be linear (which actually is automatic, check it!) and bounded. Hence if we view A in the category of bounded linear operators we call a bounded linear mapping B satisfying the conditions above the **inverse operator** to A. Also in this case we could have that A is surjective, i.e. $\mathcal{R}(A) = Y$. In particular this is natural to assume in the case $X = Y$ if we view the operator A as an element in $\mathcal{B}(X, X)$ where X is a Banach space. We observe that the space $\mathcal{B}(X, X)$, for short denoted by $\mathcal{B}(X)$, is not just a Banach space but also a Banach algebra, i.e. there is a multiplication defined in $\mathcal{B}(X)$ given by composition of operators

$$ST(x) = S(Tx)$$

which satisfies the norm inequality

$$\|ST\| \leq \|S\|\|T\|.$$

The inverse operator for A, provided A is surjective, is the inverse element to A in the Banach algebra $\mathcal{B}(X)$.

In connection with the Neumann series technique let us consider the following example. Set

$$X = Y = \mathcal{P}([0, 1])$$

and

$$Ap(x) = (1 - \frac{x}{2})p(x), \quad x \in [0, 1].$$

Moreover assume that X and Y are equipped with the L^2–norm. This means that the normed spaces $(X, \|\cdot\|_2)$ and $(Y, \|\cdot\|_2)$ are not Banach spaces. If we complete the normed space we obtain the Banach space $L^2([0, 1])$. The question is whether A is invertible or not? First we note that A is injective, i.e. $Ap = Aq$ implies $p = q$. This is straight-forward since $Ap = Aq$ in Y means that $Ap(x) = Aq(x)$ for all $x \in [0,1]$ and hence $p(x) = q(x)$ for all $x \in [0,1]$, i.e. $p = q$ in X. But A is not surjective since $\mathcal{R}(A)$ consists of all restrictions of polynomials with a zero at $x = 2$ to the interval $[0,1]$. This shows that $A : X \to \mathcal{R}(A)$ has an inverse mapping. Moreover we note that A is a bounded linear mapping from x into Y with the operator norm given by

$$\|A\| = \sup_{\substack{p\in\mathcal{P}([0,1])\\ \|p\|_2=1}} (\int_0^1 |(1-\frac{x}{2})\,p(x)|^2\,dx)^{\frac{1}{2}} = 1.$$

Prove this! A question is now if A has an inverse operator. Since A is given as a multiplication mapping it is clear that the inverse mapping also is given by a multiplication mapping where the multiplier is $\frac{2}{2-x}$. We obtain $A^{-1} : \mathcal{R}(A) \to \mathcal{P}([0,1])$ as a bounded linear mapping with the operator norm

$$\|A^{-1}\| = \sup_{\substack{p\in\mathcal{R}(A)\\ \|p\|_2=1}} (\int_0^1 |(\frac{2}{2-x})\,p(x)|^2\,dx)^{\frac{1}{2}} = 2.$$

Prove also this!

If we extend A to all of $L^2([0,1])$, call this extension $\tilde{A}$, which can be done uniquely since the polynomials in $\mathcal{P}([0,1])$ are dense in $L^2([0,1])$ and A is a bounded linear operator on $\mathcal{P}([0,1])$, we observe that $\|I - \tilde{A}\| < 1$, where $\|\cdot\|$ denote the operator norm on $L^2([0,1])$, since

$$\int_0^1 |\frac{x}{2}f(x)|^2\,dx \le \frac{1}{4}\int_0^1 |f(x)|^2\,dx,$$

and hence

$$\|(I-\tilde{A})f\| \le \frac{1}{2}\|f\|.$$

From this we get that the Neumann series $\Sigma_{n=0}^{\infty}(I-\tilde{A})^n$ gives an expression for the inverse mapping to $\tilde{A}$, since $\tilde{A}$ can be written as $\tilde{A} = I - (I - \tilde{A})$ on $L^2([0,1])$. It is no surprise that

$$\tilde{A}^{-1}p(x) = \Sigma_{n=0}^{\infty}(I-\tilde{A})^n p(x) = \Sigma_{n=0}^{\infty}(\frac{x}{2})^n p(x) = \frac{2}{2-x}p(x).$$

Observe that $(\tilde{A}|_{\mathcal{P}([0,1])})^{-1} = A^{-1}$. The Neumann series applied to an element of $\mathcal{R}(A)$ yields a polynomial, but to make sure that the series converges we need to consider the series in a Banach space and not just a normed space. Moreover we see that $\tilde{A}^{-1}$ is a bounded operator on the whole of $L^2([0,1])$ with the norm $\|\tilde{A}^{-1}\| = 2$ but A^{-1} is not a bounded operator on the whole of $\mathcal{P}([0,1])$ and this despite the fact

$$\|I - A\|_{X\to Y} \le \|I - \tilde{A}\|_{L^2\to L^2} < 1.$$

Below we present some observations that are related to the concepts inverse mapping/inverse operator.

- We first consider mappings on vector spaces. The following holds true.

  Theorem 4.1. *Assume that E is a finite-dimensional vector space and that $A : E \to E$ is a linear mapping. Then the following statements are equivalent:*

1. *A is bijective*
2. *A is injective, i.e. $\mathcal{N}(A) = \{0\}$*
3. *A is surjective, i.e. $\mathcal{R}(A) = E$*

Note that this is not true for infinite-dimensional vector spaces, which is shown by the following example.

Set

$$E = C([0, 1])$$

and

$$Af(x) = \int_0^x f(t)\, dt, \quad x \in [0, 1].$$

Prove that A is injective but not surjective!

- *From now on we only consider linear mappings $X \to Y$ where X and Y are Banach spaces.* We know that

  - *A is continuous at $x_0 \in X$ implies that A is continuous on X*
  - *A is continuous iff A is a bounded mapping.*

  It can be shown, not without some effort, that there are linear mappings $A : X \to X$ that are not bounded, i.e. the linearity and the mapping property $A(X) \subset X$ is not enough for A to be a bounded operator. This has some relevance when returning to the **stability**-question 2 in the introduction, i.e. whether the fact that $A : X \to Y$ is a bijective bounded linear operator implies that there exists a constant C such that

  $$\|f - \tilde{f}\| \leq C\|g - \tilde{g}\|,$$

  for all $g, \tilde{g} \in Y$ where $Af = g$ and $A\tilde{f} = \tilde{g}$? The answer is given by

  Theorem 4.2 (Inverse mapping theorem)**.** *Assume that $A : X \to Y$ is a bijective bounded linear mapping from the Banach space X onto the Banach space Y. Then the mapping A^{-1} exists as a bounded linear mapping from Y onto X.*

  The answer to the question above is yes!

  The proof is based on Baire's Theorem (see [4] section 1.4). Often the inverse mapping theorem is given as a corollary to the open mapping theorem, that also can be proved using Baire's theorem. We formulate the theorem without proof.

  Theorem 4.3 (Open mapping theorem)**.** *Assume that $A : X \to Y$ is a surjective bounded linear mapping from the Banach space X onto the Banach space Y. Then A maps open sets in X onto open sets in Y.*

  Recall that a mapping $A : X \to Y$ is continuous iff the set $A^{-1}(U)$ is open in X for every open set U in Y.

It has been shown, using Neumann series, that the equation $Af = g$ is uniquely solvable if $A = I - T$ and $\|T\| < 1$. However this is a serious restriction. We want to solve equations where T is not a small perturbation of the identity mapping. To do this we will, as for the finite-dimensional case, study the equation

$$(\lambda I - T)f = g$$

where λ is a complex parameter. In this context concepts like spectrum, resolvent and resolvent set are introduced. A more extensive treatment can be found in the books [2], [3] and [5]. The first two books are on the same level as the textbook.

Assume that X is a complex normed space and that $T : \mathcal{D}(T) \to X$ is a bounded linear mapping with $\mathcal{D}(T) \subseteq X$. Often we have $\mathcal{D}(T) = X$.

Definition 4.1. *The* **resolvent set** *for T, denoted $\rho(T)$, consists of all complex numbers $\lambda \in \mathbb{C}$ for which $(T - \lambda I)^{-1}$ exists as an inverse operator on all of X. The mapping $\rho(T) \ni \lambda \mapsto (\lambda I - T)^{-1}$ is called the* **resolvent** *for T.*

It follows from the definition that $\lambda \in \rho(T)$ implies that $\mathcal{N}(T - \lambda I) = \{0\}$ and that $\mathcal{R}(T - \lambda I) = X$.

Definition 4.2. *The* **spectrum** *for T, denoted by $\sigma(T)$, is the set $\mathbb{C} \setminus \rho(T)$. This set is the union of the three mutually disjoint subsets $\sigma_p(T)$, $\sigma_c(T)$ and $\sigma_r(T)$. These are called the* **point spectrum**, *the* **continuous spectrum** *and the* **residual spectrum** *respectively and are defined by the properties*

- *$\lambda \in \sigma_p(T)$ if $\mathcal{N}(T - \lambda I) \neq \{0\}$. Here λ is called an* **eigenvalue** *for T and a $v \in \mathcal{N}(T - \lambda I) \setminus \{0\}$ is called an* **eigenvector** *corresponding to the eigenvalue λ;*
- *$\lambda \in \sigma_c(T)$ if $\mathcal{N}(T - \lambda I) = \{0\}$ and $\mathcal{R}(T - \lambda I)$ is dense in X but $(T - \lambda I)^{-1}$ is not a bounded operator;*
- *$\lambda \in \sigma_r(T)$ if $\mathcal{N}(T - \lambda I) = \{0\}$ but $\mathcal{R}(T - \lambda I)$ is not dense in X.*

Examples:

1. Assume that $T : X \to X$ is a linear mapping on a finite-dimensional normed space X. Then we have $\sigma(T) = \sigma_p(T)$ and the spectrum consists of finitely many elements.
2. Consider the linear mapping $T : l^2 \to l^2$ defined by

$$(x_1, x_2, x_3, \ldots) \mapsto (0, x_1, x_2, \ldots)$$

T is a so called right shift operator. Then we have $0 \in \sigma(T) \setminus \sigma_p(T)$.

From now on we assume that T is a bounded operator.

Theorem 4.4. *The resolvent set is an open set.*

Proof. (a sketch) We note that

- if $A : X \to X$ is a bounded linear operator with $\|A\| < 1$ then $(I - A)^{-1}$ exists as an inverse operator on all of X and

  $$(I - A)^{-1} = I + A + A^2 + A^3 + \cdots$$

  (Neumann series)

- if $\lambda_0 \in \rho(T)$ we have the formula

  $$T - \lambda I = (T - \lambda_0)(I - (\lambda - \lambda_0)(T - \lambda_0 I)^{-1}).$$

Combining these observations we obtain the result. □

In this context we give a proof for the method of continuity. Condition 1 implies that all A_t, $t \in [0, 1]$, are injective. Assuming that A_t has an inverse operator defined on all of X we can write the operator A_s as

$$A_s = A_t(I + A_t^{-1}(A_s - A_t)).$$

Hence it follows that A_s is invertible if $\|A_t^{-1}(A_s - A_t)\| < 1$. But now condition 1 implies that $\|A_t^{-1}\| \le C$ and condition 2 implies $\|A_s - A_t\| \le C|s - t|$. This yields that

$$\|A_t^{-1}(A_s - A_t)\| \le \|A_t^{-1}\| \, \|A_s - A_t\| < 1$$

provided

$$|s - t| < \frac{1}{C^2}.$$

Take a finite sequence of points t_n, $0 = t_1 < t_2 < \ldots < t_n < t_{n+1} < \ldots < t_N = 1$, such that

$$\max_{n=1,2,\ldots,N-1} |t_{n+1} - t_n| < \frac{1}{C^2}.$$

The argument above shows that $A_{t_{n+1}}$ is invertible if A_{t_n} is invertible and hence the invertibility of A_0 implies the invertibility of A_1. (Invertibility of an operator B means that B^{-1} exists as an inverse operator and B is surjective.)

Theorem 4.5. *The spectrum $\sigma(T)$ belongs to the disc*

$$\{\lambda \in \mathbb{C} : |\lambda| \le \|T\|\}$$

in the complex plane.

Proof. Exercise! □

Theorem 4.6. *The spectrum $\sigma(T)$ is non-empty.*

The proof can be based on Liouville's Theorem, well-known from courses in complex analysis, but is omitted.

Definition 4.3. *The* **approximate point spectrum** *to T, denoted by $\sigma_a(T)$, consists of all $\lambda \in \mathbb{C}$ for which there exists a sequence $(x_n)_{n=1}^{\infty}$ in X, with $\|x_n\| = 1$ such that*

$$\lim_{n\to\infty} \|(T - \lambda I)x_n\| = 0.$$

The following result summarizes the important properties for the approximate point spectrum.

Theorem 4.7. *Assume that T is a bounded operator on X. Then we have:*

1. *$\sigma_a(T)$ is a closed non-empty subset of $\sigma(T)$;*
2. *$\sigma_p(T) \bigcup \sigma_c(T) \subset \sigma_a(T)$;*
3. *the boundary of $\sigma(T)$ is a subset of $\sigma_a(T)$;*

From now on we assume that the linear operator T is compact and that X is a Banach space. An operator T is called compact on X if for every bounded sequence $(x_n)_{n=1}^{\infty}$ in X there exists a convergent subsequence of $(Tx_n)_{n=1}^{\infty}$ in X. Using Riesz' Lemma (see [4] section 1.2) together with a lot of hard work one can show the following theorem that usually is called **Fredholm's alternative**.

Theorem 4.8 (Fredholm's alternative)**.** *Let T be a compact linear operator on a Banach space X and let $\lambda \in \mathbb{C} \setminus \{0\}$. Then one of the statements below hold true:*

1. *the homogeneous equation*
$$Tx - \lambda x = 0$$
has non-trivial solutions $x \in X$
2. *for every $y \in X$ the equation*
$$Tx - \lambda x = y$$
has a unique solution $x \in X$.

In the second case the operator $(T - \lambda I)^{-1}$ exists as a bounded operator.

Example: Consider the Volterra equation

$$f(x) = g(x) + \int_0^x K(x,y)f(y)\,dy \quad 0 \le x \le 1,$$

where K is a continuous function for $0 \le x, y \le 1$. Show that for every $g \in C([0,1])$ there exists a $f \in C([0,1])$ that solves the equation. From Fredholm's alternative with $X = C([0,1])$ it is enough to show that $\mathcal{N}(T-I) = \{0\}$, where T is the compact (show this using for instance Arzela-Ascoli Theorem) operator

$$Tf(x) = \int_0^x K(x,y)f(y)\,dy$$

on $C([0,1])$. We will show that

$$f(x) = \int_0^x K(x,y)f(y)\,dy \quad 0 \le x \le 1$$

implies that $f = 0$. Set $M = \max_{0\le x,y\le 1} |K(x,y)|$ and

$$\phi(x) = \int_0^x |f(y)|\,dy \quad 0 \le x \le 1.$$

It follows that ϕ is differentiable and

$$\phi^{'}(x) = |f(x)| \le M\phi(x) \quad 0 \le x \le 1$$

and hence $(\phi(x)e^{-Mx})' \le 0$ and finally

$$0 \le \phi(x) \le \phi(0)e^{-Mx} \quad 0 \le x \le 1.$$

But we have $\phi(0) = 0$ and the desired conclusion follows.

Moreover the following result holds.

Theorem 4.9 (Riesz-Schauder Theorem)**.** *Assume that $T : X \to X$ is a compact linear operator on a Banach space X. Then the following statements hold true:*

1. *$\sigma_p(T)$ is countable, can be finite or even empty;*
2. *$\lambda = 0$ is the only clustering point for the set $\sigma_p(T)$;*
3. *λ is an eigenvalue if $\lambda \in \sigma(T) \setminus \{0\}$;*
4. *X infinite-dimensional space implies that $0 \in \sigma(T)$;*
5. *For $\lambda \ne 0$ the subspaces $\mathcal{R}((T-\lambda I)^r)$ are closed and the subspaces $\mathcal{N}((T-\lambda I)^r)$ are finite-dimensional for $r = 1, 2, 3, \ldots$;*
6. *For $\lambda \ne 0$ there exists a non-negative integer r, depending on λ, such that*

$$X = \mathcal{N}((T-\lambda)^r) \bigoplus \mathcal{R}((T-\lambda)^r)$$

and

$$\mathcal{N}((T-\lambda I)^r) = \mathcal{N}((T-\lambda I)^{r+1}) = \mathcal{N}((T-\lambda I)^{r+2}) = \cdots$$

and

$$\mathcal{R}((T-\lambda I)^r) = \mathcal{R}((T-\lambda I)^{r+1}) = \mathcal{R}((T-\lambda I)^{r+2}) = \cdots .$$

Moreover if $r > 0$ it holds that

$$\mathcal{N}(I) \subset \mathcal{N}((T-\lambda I)^1) \subset \cdots \subset \mathcal{N}((T-\lambda I)^r)$$

and

$$\mathcal{R}(I) \supset \mathcal{R}((T-\lambda I)^1) \supset \cdots \supset \mathcal{R}((T-\lambda I)^r),$$

where $\subset$ and $\supset$ here denotes proper subset.

7. For $\lambda \neq 0$ it holds that[11]

$$\mathcal{R}(T - \lambda I) = \mathcal{N}(T^* - \lambda I)^\perp.$$

The last statement in the theorem has a meaning to us if X is a Hilbert space (the "Riesz part" of the theorem) but it is also possible to assign a meaning to the concept adjoint operator in a Banach space and to the "orthogonal complement" that usually is called the set of annihilators (the "Schauder part" of the theorem is the generalisation to arbitrary Banach spaces). It should be noted that the definition of adjoint operator on a Banach space differs slightly from the Hilbert space case but just up to an isometry. For those who are interested we refer to [2], [3] and [5].

If we use the last part of Riesz-Schauder's Theorem we can make Fredholm's alternative a bit more precise.

Theorem 4.10 (Fredholm's alternative)**.** *Let T be a compact linear operator on a Banach space X and let $\lambda \neq 0$. Then it holds that $Tx - \lambda x = y$ has a solution iff*[12] *$y \in \mathcal{N}(T^* - \lambda I)^\perp$.*

Now let $X = H$ be a Hilbert space and T a compact linear operator on H. If T is self-adjoint we obtain the counterpart to Fredholm's alternative that is given in the textbook [1] theorem 5.2.6, which using Hilbert space notations can be written as

$$\mathcal{R}(T - I) = \mathcal{N}(T - I)^\perp.$$

For the case with self-adjoint compact operators on Hilbert spaces the integer r in Theorem 4.9 will be equal to 1. In connection with $n \times n$–matrices and their eigenvalues this corresponds to the fact that the algebraic multiplicity and the geometric multiplicity are equal for eigenvalues to hermitian matrices.

Let us very briefly indicate the Banach space case.

For arbitrary Banach spaces X we set $X^* = \mathcal{B}(X, \mathbb{C})$, considered as a Banach space with the norm given by the operator norm $\|\cdot\|_{X\to\mathbb{C}}$. Let T be a bounded linear mapping from the Banach space X into the Banach space Y. We define the mapping $T^* : Y^* \to X^*$ using the relation

$$(T^* y^*)(x) = y^*(Tx) \quad \text{alla} \quad y \in Y^*,\ x \in X.$$

It is easy to show that T^* is a bounded linear mapping with $\|T^*\|_{Y^* \to X^*} = \|T\|_{Y \to X}$. For sets $A \subset X$ and $B \subset X^*$ in a Banach space X we set

$$A^\perp = \{x^* \in X^* : x^*(x) = 0 \quad \text{alla} \quad x \in A\}$$

and

$$B^\perp = \{x \in X : x^*(x) = 0 \quad \text{alla} \quad x^* \in A\}.$$

Here $A^\perp$ and $B^\perp$ become closed subspaces in X^* and X respectively. We detect a difference in the definition compared to the orthogonal complement for a set A in a Hilbert space! The following result can be proved (we recognise it for the case $X = \mathbb{C}^n, Y = \mathbb{C}^m$ and T given by a $m \times n$–matrix).

Theorem 4.11. *Assume that X and Y are Banach spaces and that $T \in \mathcal{B}(X, Y)$. Then it holds that*

$$\overline{\mathcal{R}(T)} = \mathcal{N}(T^*)^\perp.$$

If $\overline{\mathcal{R}(T)} = \mathcal{R}(T)$ it holds that

$$\overline{\mathcal{R}(T^*)} = \mathcal{N}(T)^\perp$$

and $\overline{\mathcal{R}(T^*)} = \mathcal{R}(T^*)$.

[11] There is a difference here compared to when X is a Hilbert space which depends on the definition of adjoint operator. If we use our standard definition from [1] the relation should be

$$\mathcal{R}(T - \lambda I) = \mathcal{N}(T^* - \overline{\lambda} I)^\perp.$$

[12] If X is a Hilbert space and the usual definition for adjoint operator is used the relation should be $y \in \mathcal{N}(T^* - \overline{\lambda} I)^\perp$.

For T in Theorem 4.11 it is true that if T is compact then T^* is also compact (the converse is also true). T being compact also implies that $\mathcal{R}(T-\lambda I)$ is closed (compare [4] section 1.6). Theorem 4.11 implies that

$$\mathcal{R}(T-\lambda I)=\mathcal{N}(T^*-\lambda I)^{\perp}$$

and

$$\mathcal{R}(T^*-\lambda I)=\mathcal{N}(T-\lambda I)^{\perp}.$$

Finally we refer to the textbook [1] for the spectral theory for compact self-adjoint operators.

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Fredholms alternativ på Banach

med Riesz sats

med lite till adjunkt annihilator

Fredholms alternativ på Hilbert

5 Ordinary differential equations

5.1 Introduction

Let $c_0, \ldots, c_n \in C(I)$ be fixed, where $I = [a, b]$, $n \geq 2$ and

$$c_n(x) \neq 0, \text{ for all } x \in I.$$

Set

$$Lu = c_n u^{(n)} + \ldots + c_0 u,\ u \in C^n(I).$$

The aim of this note is to show that the differential operator L with proper homogeneous boundary conditions has a so called Green's function. This means that the solution can be written as an integral with the Green's function appearing as the kernel function. Moreover we show that provided the operator L is symmetric the solution has a spectral decomposition. This follows from the spectral theorem for compact self-adjoint operators on Hilbert spaces ([1] Theorem 4.10.2).

5.2 Existence of Green's functions

Our first result is the following fundamental existence theorem for ordinary differential equations.

Theorem 5.1. *Assume $t_0 \in I$ and $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n$. Then for every $f \in C(I)$ there exists a unique $u \in C^n(I)$ such that $Lu = f$ and $(u(t_0), u'(t_0), \ldots, u^{(n-1)}(t_0)) = \xi$.*

Proof. Set $y_1 = u$, $y_2 = u', \ldots, y_n = u^{(n-1)}$. The equation $Lu = f$ is equivalent to

$$\begin{cases} y_1' = y_2 \\ \vdots \\ y_{n-1}' = y_n \\ \\ y_n' = -\frac{c_0}{c_n} y_1 - \ldots - \frac{c_{n-1}}{c_n} y_n + \frac{1}{c_n} f \end{cases}$$

or, using the vector notation $y = (y_1, \ldots, y_n)$,

$$y' = F(t, y),\ t \in I$$

for a vector-valued function F. This function satisfies a so called Lipschitz condition

$$|F(t, y) - F(t, z)| \leq K|y - z|,\ t \in I,\ y, z \in \mathbb{R}^n,$$

for some $K \in \mathbb{R}$. Moreover note that the condition $(u(t_0), u'(t_0), \ldots, u^{(n-1)}(t_0)) = \xi$ can be written $y(t_0) = \xi$. Picard's existence theorem ([1] theorem 5.2.5) in vector form yields the result. $\square$

We introduce the notation

$$\mathcal{N}(L) = \{u \in C^n(I); Lu = \mathbb{0}\}.$$

Clearly $\mathcal{N}(L)$ is a subspace of $C^n(I)$ since L is a linear operator.

Corollary 5.1. $\dim \mathcal{N}(L) = n$.

Proof. Let $t_0 \in I$ be fixed and define

$$Tu = (u(t_0), \ldots, u^{(n-1)}(t_0)),\ u \in \mathcal{N}(L).$$

The linear mapping $T : \mathcal{N}(L) \to \mathbb{C}^n$ is a bijection from the previous theorem with the range $\mathbb{C}^n$. Hence we get $\dim \mathcal{N}(L) = \dim \mathbb{C}^n = n$. $\square$

For arbitrary functions $u_1, \ldots, u_n \in \mathcal{N}(L)$ we define the **Wronskian** for $u_1, \ldots, u_n$ by

$$W(t) = \begin{vmatrix} u_1(t) & u_2(t) & \ldots & u_n(t) \\ u_1'(t) & u_2'(t) & & u_n'(t) \\ \vdots & \vdots & & \vdots \\ u_1^{(n-1)}(t) & u_2^{(n-1)}(t) & & u_n^{(n-1)}(t) \end{vmatrix},\ t \in I.$$

Theorem 5.2. *The following conditions are equivalent:*

1. $W(t) \neq 0$ *for all* $t \in I$.
2. $W(t_0) \neq 0$ *for some* $t_0 \in I$.
3. $u_1, \ldots, u_n$ *is a basis for the vector space* $\mathcal{N}(L)$.

Proof. (1) $\Rightarrow$ (2): trivial.

(2) $\Rightarrow$ (3): Take an $u \in \mathcal{N}(L)$. Since $\dim \mathcal{N}(L) = n$ it is enough to show that u is a linear combination of $u_1, \ldots, u_n$.

Assume that $t_0 \in I$ is fixed and that $W(t_0) \neq 0$. From courses in linear algebra we know that there exist $\alpha_1, \ldots, \alpha_n \in \mathbb{C}^n$ such that

$$\sum_{k=1}^{n} \alpha_k (u_k(t_0), \ldots, u_k^{(n-1)}(t_0)) = (u(t_0), \ldots, u^{(n-1)}(t_0)).$$

The function $v = \sum_1^n \alpha_k u_k \in \mathcal{N}(L)$ satisfies the relation

$$(v(t_0), \ldots, v^{(n-1)}(t_0)) = (u(t_0), \ldots, u^{(n-1)}(t_0))$$

and by Theorem 5.1 we have $v = u$. Hence it follows that $u \in \text{span}\ \{u_1, \ldots, u_n\}$.

(3) $\Rightarrow$ (1): Let $t \in I$ be arbitrary. We will show that $W(t) \neq 0$. It is enough to show that the columns in the determinant $W(t)$ are linearly independent.

Assume that $\alpha_1, \ldots, \alpha_n \in \mathbb{C}^n$ and that

$$\sum_{k=1}^{n} \alpha_k (u_k(t), \ldots, u_k^{(n-1)}(t)) = (0, 0, \ldots, 0).$$

The function $v = \sum_1^n \alpha_k u_k \in \mathcal{N}(L)$ satisfies $v(t) = \ldots = v^{(n-1)}(t) = 0$ and is equal to the zero function by Theorem 5.1. However from $\sum_1^n \alpha_k u_k = \mathbb{0}$ it follows that $\alpha_1 = \ldots = \alpha_n = 0$. Hence the columns in the determinant $W(t)$ are linearly independent. $\square$

From now on we use the following notation:

$$\alpha_{ij}, \beta_{ij}, i = 0, \dots, n-1,\ j = 1, \dots, n$$

are complex numbers and

$$R_j u = \sum_{i=0}^{n-1} [\alpha_{ij} u^{(i)}(a) + \beta_{ij} u^{(i)}(b)],\ j = 1, \dots, n.$$

are boundary operators. Moreover we set

$$Ru = (R_1 u, \dots, R_n u)$$

$$C_R^n(I) = \{u \in C^n(I) : Ru = 0\}$$

and

$$L_0 u = Lu,\ u \in C_R^n(I).$$

Theorem 5.3. *The following conditions are equivalent:*

1. *The mapping $L_0 : C_R^n(I) \to C(I)$ is a bijection.*
2. $\det\{R_j u_k\}_{1 \le j,k \le n} \neq 0$ *for every (alternatively for some) basis $u_1, \dots, u_n$ i $\mathcal{N}(L)$.*

Proof. (1) ⇒ (2): If the determinant in (2) is zero then there are $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ not all equal to zero such that

$$\sum_{k=1}^{n} \alpha_k R_j u_k = 0,\ j = 1, \dots, n.$$

The function $v = \sum_1^n \alpha_k u_k$ satisfies $Lv = \mathbb{0}$ together with $Rv = 0$. This yields a contradiction since $v \neq 0$ and $L_0 v = \mathbb{0}$.

(2) ⇒ (1): Take an arbitrary $f \in C(I)$. It remains to prove that the equation

$$\begin{cases} Lu = f \\ Ru = 0 \end{cases}$$

is uniquely solvable. Set $w = u - v$, where $v \in C^n(I)$ satisfies $Lv = f$ (Theorem 5.1), we obtain the equivalent equation

$$\begin{cases} Lw = \mathbb{0} \\ Rw = -Rv. \end{cases}$$

With the ansatz $w = \sum_1^n \alpha_k u_k$ the determinant condition in (2) gives the existence of a unique solution. □

Now let $u_1, \dots, u_n$ be a basis for the vector space $\mathcal{N}(L)$ and set

$$e(x,t) = \sum_{k=1}^{n} a_k(t) u_k(x)$$

where $a_1(t), \ldots, a_n(t)$ are chosen such that

$$\begin{cases} e_x^{(k)}(t,t) = 0, \; k = 0, 1, \ldots, n-2 \\ e_x^{(n-1)}(t,t) = 1/c_n(t). \end{cases}$$

Note that the functions $a_1(t), \ldots, a_n(t)$ are continuous in t due to Cramer's rule. Also observe that for fixed $t \in I$ the function $u(x) = e(x,t)$ is the unique solution to the equation

$$\begin{cases} Lu = \mathbb{0} \\ u(t) = \ldots = u^{(n-2)}(t) = 0, \; u^{(n-1)}(t) = 1/c_n(t). \end{cases}$$

The function $e(x,t), (x,t) \in I \times I$, is called the **fundamental solution** to the operator L. This function is of interest in connection with boundary value problems that we will discuss next.

Theorem 5.4. *Let $u_1, \ldots, u_n$ be a basis for $\mathcal{N}(L)$ such that*

$$\det\{R_j u_k\}_{1 \le j,k \le n} \neq 0$$

and set $G = L_0^{-1}$. Then there exists a unique continuous function $g(x,t), (x,t) \in I \times I$, such that

$$(Gf)(x) = \int_I g(x,t) f(t) dt.$$

This is called the Green's function g and can be constructed as follows:

1. *Set $\tilde{e}(x,t) = \theta(x-t)e(x,t)$, where θ is the Heaviside's function and $e(x,t)$ is the fundamental solution to L*
2. *Determine $b_1, \ldots, b_n \in C(I)$ such that the function*

$$g(x,t) = \tilde{e}(x,t) + \sum_{k=1}^{n} b_k(t) u_k(x)$$

   satisfies

$$R(g(\cdot,t)) = 0, \; a < t < b.$$

Proof. First set

$$\tilde{u}(x) = \int_I \tilde{e}(x,t) f(t) dt,$$

i.e.

$$\tilde{u}(x) = \int_a^x e(x,t) f(t) dt.$$

Repeated differentiations yield

$$\begin{aligned} \tilde{u}'(x) &= \int_a^x e_x'(x,t) f(t) dt + \underbrace{e(x,x)}_{=0} f(x) \\ \tilde{u}''(x) &= \int_a^x e_x''(x,t) f(t) dt + \underbrace{e_x'(x,x)}_{=0} f(x) \\ &\vdots \\ \tilde{u}^{(n-1)}(x) &= \int_a^x e_x^{(n-1)}(x,t) f(t) dt + \underbrace{e^{(n-2)}(x,x)}_{=0} f(x) \end{aligned}$$

and

$$\tilde{u}^{(n)}(x) = \int_a^x e^{(n)}(x,t)f(t)dt + \frac{1}{c_n(x)}f(x).$$

From this we conclude $L\tilde{u} = f$. The function

$$u(x) = \int_I g(x,t)f(t)dt$$

satisfies the equation $Lu = f$ since

$$u(x) = \tilde{u}(x) + \sum_{k=1}^{n} u_k(x) \int_I b_k(t)f(t)dt.$$

Finally we observe that

$$Ru = \int_{a^+}^{b^-} \underbrace{R(g(\cdot,t))}_{=0} f(t)dt$$

which proves the existence of a continuous Green´s function. To show uniqueness assume the existence of two Green´s functions $g(x,t)$, $\tilde{g}(x,t)$. Then

$$\int_a^b (g(x,t) - \tilde{g}(x,t))f(t)\,dt = 0, \quad x \in [a,b]$$

holds for all $f \in C([a,b])$ and the conclusion follows. $\square$

The function g in Theorem 5.4 is called the **Green's function for the boundary value problem**

$$\begin{cases} Lu = f \\ Ru = 0. \end{cases}$$

Problem 1: Determine the Green's function for the boundary value problem

$$\begin{cases} -((1+x)u'(x))' = f(x),\ 0 \le x \le 1 \\ u'(0) = 0,\ u(1) = 0. \end{cases}$$

Solution: The functions $u_1(x) = 1$ and $u_2(x) = \ln(1+x)$ form a basis for the solutions to the homogeneous equation $-((1+x)u'(x))' = 0$. Note that

$$\begin{vmatrix} u_1'(0) & u_2'(0) \\ u_1(1) & u_2(1) \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & \ln 2 \end{vmatrix} = -1 \neq 0.$$

so there exists a Green's function. The fundamental solution $e(x,t) = a_1(t)u_1(x) + a_2(t)u_2(x)$ is given by

$$e(x,t) = a_1(t) + a_2(t)\ln(1+x)$$

and the constraints $e(t,t) = 0$, $e'_x(t,t) = -\frac{1}{1+t}$ easily yield

$$e(x,t) = \ln(1+t) - \ln(1+x).$$

The Green's function takes the form

$$g(x,t) = \theta(x-t)(\ln(1+t) - \ln(1+x)) + b_1(t) + b_2(t)\ln(1+x)$$

where

$$\begin{cases} g'_x(0,t) = 0 \\ g(1,t) = 0, \end{cases}$$

for $0 < t < 1$. Hence we get

$$\begin{cases} b_2(t) = 0 \\ \ln(1+t) - \ln 2 + b_1(t) + b_2(t)\ln 2 = 0 \end{cases}$$

from which we obtain

$$b_1(t) = \ln\frac{2}{1+t},\ b_2(t) = 0.$$

This finally gives

$$g(x,t) = \theta(x-t)\ln\frac{1+t}{1+x} + \ln\frac{2}{1+t}.$$

Problem 2: Assume that $\lambda \in \mathbb{C}$ and $f \in C([0,1])$. Show that the equation

$$\begin{cases} u''(x) + u'(x) + \lambda|u(x)| = f(x),\ 0 \le x \le 1 \\ u(0) = u(1) = 0,\ u \in C^2([0,1]) \end{cases}$$

has a unique solution for $|\lambda| < e(e-1)$.

Solution: We first determine the Green's function for the equation

$$\begin{cases} u'' + u' = F(x),\ 0 \le x \le 1 \\ u(0) = u(1) = 0. \end{cases}$$

The functions $u_1(x) = 1$ and $u_2(x) = e^{-x}$ form a basis for the solutions to the homogeneous equation $u'' + u' = 0$. With our standard notation we get

$$e(x,t) = 1 - e^{t-x}$$

and

$$g(x,t) = \theta(x-t)(1-e^{t-x}) + \frac{e^t - e}{e-1} + \frac{e-e^t}{e-1}e^{-x}.$$

Note that

$$t > x \Rightarrow g(x,t) = \frac{e^t - e}{e-1}(1-e^{-x}) \le 0$$

and

$$t \le x \Rightarrow g(x,t) = \frac{e^t - 1}{e-1}(1-e^{1-x}) \le 0$$

which implies $g \le 0$.

For every $u \in C([0,1])$ define

$$(Tu)(x) = \int_0^1 g(x,t)(f(t) - \lambda|u(t)|)dt,\ 0 \le x \le 1$$

and observe that T maps $C([0,1])$ into $\{u \in C^2([0,1]);\ u(0) = u(1) = 0\}$. The equation in problem 2 has therefore a unique solution iff T has a unique fixed point. For $u, v \in C([0,1])$ it holds that

$$|(Tu)(x) - (Tv)(x)| = |\int_0^1 g(x,t)(\lambda|v(t)| - \lambda|u(t)|)dt| \leq$$
$$\leq |\lambda| \int_0^1 (-g(x,t))||v(t)| - |u(t)||dt \leq |\lambda| j(x) \|u - v\|_\infty,$$

where $\| \|_\infty$ denotes the max-norm for $C([0,1])$ and

$$j(x) = -\int_0^1 g(x,t)dt.$$

Since $j(0) = j(1) = 0$ and $j'' + j' = -1$ it follows that

$$j(x) = \frac{e}{e-1} - x - \frac{e}{e-1}e^{-x}$$

and

$$\max_{[0,1]} j = j\left(\ln \frac{e}{e-1}\right) = \frac{1}{e-1} + \ln\left(1 - \frac{1}{e}\right) \leq$$
$$\leq \frac{1}{e-1} - \frac{1}{e} = \frac{1}{e(e-1)}.$$

We conclude that

$$\|Tu - Tv\|_\infty \leq \frac{|\lambda|}{e(e-1)} \|u - v\|_\infty$$

and Banach's fixed point theorem ([2]) implies that T has a unique fixed point for $|\lambda| < e(e-1)$.

5.3 Spectral theory for ordinary differential equations

The linear mapping $L_0 : C^n_R(I) \to C(I)$ is called **symmetric** if

$$\langle L_0 u, v\rangle = \langle u, L_0 v\rangle, \text{ all } u, v \in C^n_R(I),$$

where the inner product is given by the inner product in $L^2(I)$

$$\langle f, h\rangle = \int_a^b f(x)\overline{h(x)}dx.$$

Provided that L_0 is a bijection and g is the Green's function for the boundary value problem

$$\begin{cases} Lu = f \\ Ru = 0 \end{cases},$$

we define

$$(Gf)(x) = \int_a^b g(x,t)f(t)dt,\ f \in C(I)$$

and

$$(\tilde{G}f)(x) = \int_a^b g(x,t)f(t)\,dt,\ f \in L^2(I).$$

Theorem 5.5. *Assume that L_0 is a bijection. Then the following conditions are equivalent:*

1. *L_0 is symmetric*
2. *$\tilde{G}$ is self-adjoint*
3. *$g(x,t) = \overline{g(t,x)}$, $x, t \in I$.*

Proof. (1) $\Leftrightarrow$ (2): L_0 is symmetric iff

$$\langle L_0 Gf, Gh \rangle = \langle Gf, L_0 Gh \rangle, \; f, h \in C(I)$$

which is the same as

$$\langle f, Gh \rangle = \langle Gf, h \rangle, \; f, h \in C(I).$$

This is equivalent to

$$\langle f, \tilde{G}h \rangle = \langle \tilde{G}f, h \rangle, \; f, h \in L^2(I)$$

since $C(I)$ is dense in $L^2(I)$ and $\tilde{G}$ is a bounded linear operator on $L^2(I)$ ([1] example 4.2.4) whose restriction to $C(I)$ is equal to G. L_0 being symmetric is thus equivalent to $\tilde{G}$ being self-adjoint.

(2) $\Leftrightarrow$ (3): We first observe that

$$(\tilde{G}^* f)(x) = \int_a^b \overline{g(t,x)} f(t) dt$$

([1] example 4.4.6). This implies that $\tilde{G} = \tilde{G}^*$ iff

$$\int_a^b (g(x,t) - \overline{g(t,x)}) f(t) dt = 0, \; f \in L^2(I).$$

Since g is continuous this means that $g(x,t) - \overline{g(t,x)} = 0$ for all $x, t \in I$ and so $g(x,t) = \overline{g(t,x)}$ for all $x, t \in I$. $\square$

Example 1: Consider the boundary value problem

$$\begin{cases} -u'' = f(x) \\ u(0) = u(1) = 0, \; 0 \le x \le 1. \end{cases}$$

This means that $Lu = -u''$, $R_1 u = u(0)$ and $R_2 u = u(1)$. The operator L_0 is symmetric since

$$\langle L_0 u, v \rangle = \int_0^1 -u'' \bar{v} dx = \big[-u' \bar{v} \big]_0^1 + \int_0^1 u' \bar{v}' dx = \{Rv = 0\} =$$

$$= \langle u', v' \rangle = \overline{\langle v', u' \rangle} = \{Ru = 0\} = \overline{\langle L_0 v, u \rangle} = \langle u, L_0 v \rangle$$

for all $u, v \in C_R^2([0,1])$. This fact also follows from Theorem 5.5 by checking that L_0 is a bijection and that the Green's function is given by

$$g(x,t) = \begin{cases} t(1-x), & 0 \le t < x \le 1 \\ (1-t)x, & 0 \le x \le t \le 1. \end{cases}$$

It easily follows that $g(x,t) = \overline{g(t,x)}$. The details are left as an exercise.

Theorem 5.6. *Assume that L_0 is symmetric and is a bijection. Then the following statements are true:*

1. *0 is not an eigenvalue for L_0 nor for $\tilde{G}$.*

2. *f is an eigenfunction for L_0 corresponding to the eigenvalue μ iff f is an eigenfunction for $\tilde{G}$ corresponding to the eigenvalue $1/\mu$.*

Proof. (1): $\mathcal{N}(L_0) = \{\mathbb{0}\}$ implies that L_0 has no eigenfunction corresponding to an eigenvalue zero.

Now assume that $f \in \mathcal{N}(\tilde{G})$. We will show that $f = \mathbb{0}$. For this take an arbitrary $\phi \in C_R^n(I)$. We obtain

$$\begin{aligned} 0 =& \langle \mathbb{0}, L_0\phi \rangle = \langle \tilde{G}f, L_0\phi \rangle = \langle f, \tilde{G}L_0\phi \rangle = \\ =& \langle f, GL_0\phi \rangle = \langle f, \phi \rangle. \end{aligned}$$

Since $C_R^n(I)$ is dense in $L^2(I)$ we can conclude that $f = \mathbb{0}$.

(2): $\Rightarrow$) From

$$\mathbb{0} \neq f = G(L_0 f) = G(\mu f) = \mu G f = \mu \tilde{G} f$$

it follows that f is an eigenfunction to $\tilde{G}$ corresponding to the eigenvalue $1/\mu$.

$\Leftarrow$) We have

$$\int_a^b g(x,t) f(t) dt = \frac{1}{\mu} f(x) \quad \text{a.e. in } I.$$

Setting

$$h(x) = \mu \int_a^b g(x,t) f(t) dt,\ x \in I$$

it follows from Lebesgue's dominated convergence theorem (see [3]) that $h \in C(I)$. Moreover we have $h(x) = f(x)$ *a.e.* in I and

$$h(x) = \mu \int_a^b g(x,t) h(t) dt,\ x \in I,$$

and hence we get $Gh = \frac{1}{\mu} h$. This yields

$$h = L_0(Gh) = L_0\left(\frac{1}{\mu} h\right) = \frac{1}{\mu} L_0 h.$$

Since $h \neq \mathbb{0}$ in $C_R^n(I)$, h is an eigenfunction to L_0 corresponding to the eigenvalue μ. Thus h, which is equal to f in $L^2(I)$, is an eigenfunction to L_0 corresponding to the eigenvalue μ. This is the proper interpretation of the formulation in Theorem 5.6 2) and the proof of the theorem is complete. $\square$

Theorem 5.7. *Assume that L_0 is symmetric and is a bijection. Moreover let $(\mu_n)_1^\infty$ denote the eigenvalues for L_0 counted with multiplicity and assume that $(e_n)_1^\infty$ is a corresponding sequence* *of orthonormal eigenfunctions. Then* $(e_n)_1^\infty$ *is an ON-basis for* $L^2(I)$ *and the solution to the equation*

$$\begin{cases} Lu = f \\ Ru = 0 \end{cases},$$

where $f \in C(I)$*, is given by*

$$u = \sum_1^\infty \frac{1}{\mu_n}\langle f, e_n\rangle e_n \quad (\textit{in } L^2(I)).$$

Proof. The operator $\tilde{G}$ is compact ([1] example 4.8.4) and the Hilbert-Schmidt theorem ([1] theorem 4.10.1) and Theorem 5.6 1) implies that $(e_n)_1^\infty$ is a complete ON-sequence for $L^2(I)$. (This also shows that $L^2(I)$ is a separable Hilbert space.) From

$$f = \sum_1^\infty \langle f, e_n\rangle e_n$$

in $L^2(I)$, Theorem 5.6 2) now implies that

$$u = Gf = \tilde{G}f = \sum_1^\infty \langle f, e_n\rangle \tilde{G}e_n = \sum_1^\infty \frac{1}{\mu_n}\langle f, e_n\rangle e_n$$

in $L^2(I)$. $\square$

Example 2: Consider the boundary value problem

$$\begin{cases} -u'' = f(x) \\ u(0) = u(1) = 0,\ 0 \le x \le 1. \end{cases}$$

Example 1 shows that the corresponding operator L_0 is symmetric and is a bijection. The eigenfunctions for L_0 are obtained as the non-trivial solutions to the equation

$$\begin{cases} -e''(x) = \mu e(x) \\ e(0) = e(1) = 0,\ 0 \le x \le 1 \end{cases}$$

and a simple calculation gives $e_n(x) = A\sin n\pi x$, where $A \neq 0$ and $n = 1, 2, \ldots$. The sequence $(\sqrt{2}\sin n\pi x)_1^\infty$ is therefore an ON-basis for $L^2([0,1])$.

Example 3: Wirtinger's inequality states that

$$\int_0^1 |u'(x)|^2 dx \ge \pi^2 \int_0^1 |u(x)|^2 dx$$

for all $u \in C^1([0,1])$ that satisfies $u(0) = u(1) = 0$. To show this we first let

$$u(x) = \sum_1^\infty a_n \sqrt{2}\sin n\pi x \quad (\text{in } L^2([0,1]))$$

where

$$a_n = \int_0^1 u(x)\sqrt{2}\sin n\pi x dx.$$

Furthermore we have

$$\int_0^1 u'(x)\sqrt{2}\cos n\pi x dx = \Big[u(x)\sqrt{2}\cos n\pi x\Big]_0^1 +$$
$$+ n\pi \int_0^1 u(x)\sqrt{2}\sin n\pi x dx = n\pi a_n$$

and using the fact that the sequence $(\sqrt{2}\cos n\pi x)_1^\infty$ is an ON sequence, Bessel's inequality ([1] theorem 3.4.9) yields the estimate

$$\int_0^1 |u'(x)|^2 dx \geq \sum_1^\infty n^2\pi^2 |a_n|^2$$

where the RHS is greater than or equal to

$$\pi^2 \sum_1^\infty |a_n|^2 = \pi^2 \int_0^1 |u(x)|^2 dx.$$

This gives one proof for Wirtinger's inequality.

References

[1] L.Debnath/P.Mikusinski, *Introduction to Hilbert Spaces with Applications 3rd ed.*, Academic Press 2005

[2] P.Kumlin, *A note on fixed point theory*, Mathematics, Chalmers & GU 2018/2019

[3] P.Kumlin, *A note on L^p–spaces*, Mathematics, Chalmers & GU 2018/2019

6 Exercises

This is a collection of problems that has appeared in the course. Some of them has been given on written examinations during the last years.

6.1 Vector spaces

Key words: vector space, linear combination, linear independence, basis, dimension

1. Check if the following sets with the proposed addition $\oplus$ and multiplication by scalar $\odot$ defines vector spaces:

   (a) $E = \mathbb{R}_+ \equiv \{x \in \mathbb{R} : x > 0\}$ and $F = \mathbb{R}$ with
   $$x \oplus y = xy \text{ for all } x, y \in E$$
   and
   $$\alpha \odot x = x^\alpha \text{ for all } \alpha \in F,\, x \in E.$$

   (b) $E = \mathbb{C}$ and $F = \mathbb{C}$ with
   $$x \oplus y = x + y \text{ for all } x, y \in E$$
   and
   $$\alpha x = (\operatorname{Re} \alpha) x \text{ for all } \alpha \in F,\, x \in E.$$

2. Let x be an element of a vector space and λ a scalar. Show that

   (a) $0x = \mathbf{0}$

   (b) $(-1)x = -x$

   (c) $\lambda \neq 0$ and $\lambda x = \mathbf{0}$ implies $x = \mathbf{0}$

   (d) $x \neq 0$ and $\lambda x = \mathbf{0}$ implies $\lambda = 0$

3. Let E be a vector space such that there exist a basis with finitely many vectors. Show that the dimension of E is uniquely defined.

4. Let $x_1, \ldots, x_n$ be a basis for a complex vector space E. Find a basis for E as a real vector space.

5. Let $x_1, \ldots, x_n$ be a set of linearly dependent vectors in a complex vector space E. Is this set linearly dependent in E if E is regarded as a real vector space?

6. Show that the functions $f_n(x) = e^{nx}$, $n = 1, 2, \ldots$, defined on $\mathbb{R}$ are linearly independent.

7. Show that the functions $f_n(x) = \cos nx$, $n = 1, 2, \ldots$, defined on $[-\pi, \pi]$ are linearly independent.

8. Show that the vectors $\mathbb{x}_\alpha = (1, \alpha, \alpha^2, \alpha^3, \ldots)$, $\alpha \in (0, 1)$, in l^1 are linearly independent.

9. In $C[-1,1]$ consider the sets U and V consisting of odd and even functions in $C[-1,1]$ respectively. Show that U and V are subspaces and that $U \bigcap V = \{0\}$. Show that every $f \in C[-1,1]$ can be written in the form $f = f_1 + f_2$, where $f_1 \in U$ and $f_2 \in V$, and that this decomposition is unique.

10. Let $E = C([0,1])$. Show that

    (a) if a_k, $k = 1, \ldots, n$ are n distinct points in $[0,1]$ then the functions

    $$x \mapsto |x - a_k|, \quad k = 1, \ldots, n$$

    are linearly independent on E,

    (b) the function

    $$(x,y) \mapsto |x-y|$$

    on $[0,1] \times [0,1]$ cannot be written as a finite sum

    $$\Sigma_{i=1}^n v_i(x) w_i(y),$$

    where $v_i, w_i \in E$, $i = 1, \ldots, n$.

11. Prove that the vector space $C([0,1])$ has infinite dimension.

12. Prove that the vector space $C^\infty(\mathbb{R})$ has infinite dimension.

13. Prove that the vector spaces l^p are infinite-dimensional for $p \in [1, \infty)$.

14. Let l^0 consist of all sequences $(x_n)_{n=1}^\infty$, $x_n \in \mathbb{R}$, where at most finitely many x_n:s are different from 0. Show that l^0 is a vector space with the usual addition and multiplication with scalar operations for sequence spaces. Also give a basis for l^0.

15. Let F be a subspace of a vector space E. The **coset** of an element $x \in E$ with respect to F is denoted by $x + F$ and is defined to be the set

    $$x + F = \{x + y : y \in F\}.$$

    Show that under the algebraic operations

    $$(x+F) + (y+F) = (x+y) + F$$

    $$\alpha(x+F) = \alpha x + F$$

    these cosets constitute the elements of a vector space. This vector space is called the **quotient space of E by F** and is denoted by E/F. Its dimension is called the **codimension** of F and is denoted by $\operatorname{codim} F$. Now let $E = \mathbb{R}^3$ and $F = \{(0,0,z) : z \in \mathbb{R}\}$. Find

    (a) E/F
    (b) E/E
    (c) $E/\{0\}$

16. Show that $C([c,d])$ is a subspace of $C([a,b])$ (in a natural way) if $[c,d] \subset [a,b]$.

17. Assume M and N are subspaces of a vector space V. When is $M \bigcup N$ a subspace?

18. Let $T : E \to F$ be a linear mapping from the vector space E into the vector space F. Show that $\mathcal{N}(T)$ and $\mathcal{R}(T)$ are vector spaces.

19. Show that linear mappings preserve linear dependence.

20. Let T be a linear bijection between two vector spaces E and F. Assume that E is finite-dimensional. Show that also F is finite-dimensional and that $\dim E = \dim F$.

21. The **convex hull** $\hat{S}$ of a set S is defined as the intersection of all convex sets containing S.

    (a) Show that $\hat{S}$ is convex.

    (b) If $S \subset R$ and R convex, show that $\hat{S} \subset R$.

    (c) A **convex combination** of elements $x_1, \ldots, x_n$ of a vector space is a linear combination $\Sigma a_i x_i$ with $a_i \geq 0$ for each i and $\Sigma a_i = 1$. If R is a convex set, show that any convex combination of a finite number of elements of R belongs to R.

    (d) Show that for any set S, $\hat{S}$ equals the set of all convex combinations of finitely many elements of S.

6.2 Normed spaces

Key words: norm, convergence in normed space, equivalence of norms, open/closed ball, open/closed set, closure of set, dense subset, compact set

1. Show that in any normed space

   (a) a convergent sequence has a unique limit;

   (b) if $x_n \to x$ and $y_n \to y$ then $x_n + y_n \to x + y$;

   (c) if $x_n \to x$ and $\lambda_n \to \lambda$ (λ_n, λ are scalars) then $\lambda_n x_n \to \lambda x$.

2. Let E be a normed space. Prove that

$$\|x\| \leq \max(\|x - y\|, \|x + y\|),\ x, y \in E.$$

   Give an example of a normed space E and an $x \in E$, such that equality occurs for a suitable $y \neq 0$.

3. Let X be a vector space and let $\|x\|$ and $\|x\|_*$, $x \in X$, be two norms on X. Is $\max(\|x\|, \|x\|_*)$ a norm on X? Is $\min(\|x\|, \|x\|_*)$ a norm on X?

4. Let $x_1, \ldots, x_n$ be linearly independent vectors in a normed space E. Show that there exists a $c > 0$ such that

$$\|\alpha_1 x_1 + \ldots + \alpha_n x_n\| \geq c(|\alpha_1| + \ldots + |\alpha_n|),$$

   for all scalars $\alpha_i,\ 1 \leq i \leq n$. Conclude from this that any two norms on E are equivalent, if E is finite dimensional.

5. Show that equivalent norms define the same opens sets and Cauchy sequences.

6. Show that the norms $\|\ \|_1$ and $\|\ \|_\infty$ are not equivalent in the vector space $C([0, 1])$ where

$$\|f\|_1 = \int_0^1 |f(t)|\, dt$$

   and

$$\|f\|_\infty = \max_{t \in [0,1]} |f(t)|$$

   for $f \in C([0, 1])$.

7. Given a set X. A function $d : X \times X \to [0, \infty)$ is called a **metric** on X if d satisfies the conditions

   (a) $d(x, y) = 0$ iff $x = y$

   (b) $d(x, y) = d(y, x)$ for all $x, y \in X$

   (c) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$

Show that if E is a vector space with norm $\|\cdot\|$ then

$$d(x,y) = \|x-y\| \quad x,y \in E$$

defines a metric on E.

8. Let (X,d) be a metric space. Show that d_1 given by

$$d_1(x,y) = \frac{d(x,y)}{1+d(x,y)} \quad \text{for } \ x,y \in X$$

is a metric on X. Show that the metrics d and d_1 yield the same open sets.

9. Give an example of a metric on a vector space that is not given by a norm.

10. Show that the open balls $B(x,r)$ in a normed space are open sets. Also show that the closed balls are closed sets.

11. A subset A of a vector space E is called **convex** if

$$\alpha x + (1-\alpha)y \in A \ \text{ for all } \ x,y \in A, \ \alpha \in [0,1].$$

If E is a normed space show that the closed and open unit balls $\bar{B}(0,1)$ and $B(0,1)$ are convex.

12. Set $\phi : \mathbb{R}^2 \to [0,\infty)$, where

$$\phi(x,y) = (\sqrt{|x|} + \sqrt{|y|})^2.$$

Show that ϕ does not define a norm in $\mathbb{R}^2$.

13. Let U be a bounded open convex and symmetric (i.e. $U = (-1)U$) set in $\mathbb{R}^2$ containing the origin and set

$$\|(x,y)\| = \inf\{\lambda > 0 : (x,y) \in \lambda U\},$$

where $\lambda U = \{(\lambda x, \lambda y) : (x,y) \in U\}$ for $\lambda \in \mathbb{R}$. Show that $\| \ \|$ defines a norm on $\mathbb{R}^2$. Conclude that all norms on $\mathbb{R}^2$ are given in this way.

14. Find a sequence $(x_1, x_2, \ldots)$ such that $x_n \to 0$ as $n \to \infty$ but is not in any l^p, where $1 \leq p < \infty$. Find a sequence $(x_1, x_2, \ldots)$ which is in l^p with $p > 1$ but not in l^1. Is $l^p \setminus l^q = \emptyset$ if $p > q$?

15. Give an example of a subspace in l^2 that is not closed.

16. Let $1 \leq r < p < 2r$ and assume that the sequence $(x_1, x_2, \ldots)$ satisfies

$$\Sigma_{n=1}^{\infty} n|x_n|^p < \infty.$$

Show that $(x_1, x_2, \ldots) \in l^r$.

17. Show that

$$\lim_{j\to\infty} \Sigma_{n=1}^{\infty} \frac{x_n}{j+n} = 0$$

for all $(x_1, x_2, \ldots) \in l^2$.

18. Let $f(x) = \sin x$ for $0 \le x \le 1$. Find a sequence of polynomials $p_n(x),\ 0 \le x \le 1,\ n \in \mathbb{N}$ of degree n, which converges to f in $C([0,1])$.

19. Show that every continuous function f on $[0,1]$ can be uniformly approximated by polynomials, i.e. for each $\epsilon > 0$ there is a polynomial p such that $\max_{t\in[0,1]} |f(t) - p(t)| < \epsilon$. This statement is known as the **Weierstrass approximation theorem**[13].

20. Show that if A is dense in B and B is dense in C then A is dense in C.

21. Prove or disprove: if A is dense in B then for any set C, $A \bigcap C$ is dense in $B \bigcap C$.

22. Let E be a normed space. E is called **separable** if there exists a countable dense subset in E. Show that

    (a) $\mathbb{R}$ is separable

    (b) l^p is separable for $p \in [1, \infty)$

    (c) l^∞ is not separable[14]

    (d) $C([0,1])$ is separable

23. Let E be a normed space and $(x_n)_{n=1}^\infty$ a countable dense subset in E. Given $\epsilon > 0$ show that

$$E \setminus \{0\} \subset \bigcup_{n=1}^{\infty} B(x_n, \epsilon \|x_n\|).$$

24. Show that every finite set is compact.

25. Show that $\mathbb{R}^n$ and $\bar{B}(0,1) \bigcap \{(x_1, \dots, x_n) : x_1 < 1/2\}$ are not compact sets using the definition of compactness.

26. Construct a set in $\mathbb{R}^2$ which has finite area but is not relatively compact. Generalize to $\mathbb{R}^n$.

27. Prove that any finite-dimensional subspace of a normed linear space is closed.

28. If S is a relatively compact set, prove that its convex hull is relatively compact.

29. Let F be a subspace of a normed space E and suppose $x_0 \in E \setminus F$. Furthermore suppose x_0 possesses a nearest point in F (i.e. there is a $y_0 \in F$ such that $\|y - x_0\| \ge \|y_0 - x_0\|$ for all $y \in F$).

    (a) Prove that there is an $x_1 \in E$ such that $\|x_1\| = 1$ and $\|y - x_1\| \ge 1$ for all $y \in F$.

    (b) In addition, suppose $\mathrm{Span}(\{x_0\} \bigcup F) = E$. Show that every $x \in E$ possesses a nearest point in F.

[13]Hint: One way to prove the claim is to use the so called Bernstein polynomials, more precisely set

$$B_n f(x) = \Sigma_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} x^k (1-x)^{n-k} f(\frac{k}{n}),\ x \in [0,1],\ n = 1, 2, \dots$$

Show that $B_n f \to f$ in $C([0,1])$ as $n \to \infty$.

[14]Assume that it is separable and construct a function that has l^∞-distance ≥ 1 to each function in the supposed countable dense set.

30. (**Riesz lemma**) Suppose E is a normed space and let F be a proper closed subspace of E. Furthermore let ϵ be a given positive real number. Show that there is a vector $x_1 \in E$ such that $\|x_1\| = 1$ and $\|y - x_1\| > 1 - \epsilon$ for every $y \in F$.

31. Let E be a normed space. Show that the unit sphere $\{x \in E; \|x\| = 1\}$ is compact if and only if E is of finite dimension.

32. Let F be a closed subspace of a normed space E, where $\|\cdot\|$ denotes the norm. Show that $\|\cdot\|_0$ defines a norm on the quotient space E/F if

$$\|\tilde{x}\|_0 = \inf_{x \in \tilde{x}} \|x\|.$$

33. Let T be a mapping on a real normed space X satisfying

$$T(x+y) = T(x) + T(y) \text{ for all } x, y \in X.$$

Show that

$$T(\lambda x) = \lambda T(x) \text{ for all } \lambda \in \mathbf{R} \text{ and } x \in X$$

if T is continuous.

34. Let $T : X \to X$ be a mapping (not necessary linear) on a normed space X. Moreover assume that there are real constants C, α, where $\alpha > 1$, such that

$$\|T(x) - T(y)\| \leq C\|x - y\|^\alpha, \text{ for all } x, y \in X.$$

Show that there exists a $z \in X$ such that $T(x) = z$ for all $x \in X$.

6.3 Banach spaces

Key words: Cauchy sequence, complete space, Banach space, convergent/absolutely convergent series, linear mapping, null space of a linear mapping, range and graph of a mapping, continuous mapping, bounded linear mapping, completion of a normed space, L^p-spaces

1. Prove that convergence in $L^2([0,1])$ implies convergence in $L^1([0,1])$.

2. For any $n \in \mathbb{Z}_+$ set
$$f_n(x) = \begin{cases} \sqrt{n} & 0 \le x \le \frac{1}{n} \\ 0 & \frac{1}{n} < x \le 1. \end{cases}$$
Prove that $f_n \to 0$ in $L^1([0,1])$ but not in $L^2([0,1])$.

3. Let $f \in L^1(\mathbb{R})$. Can we conclude that $f(x) \to 0$ for $|x| \to \infty$? Can we find $a, b \in \mathbb{R}$ such that $|f(x)| \le b$ for $|x| \ge a$?

4. Which of the following sequences of real functions ($n \in \mathbb{N}$)
   (a) $f_n = \frac{1}{n}\chi_{(0,n)}$
   (b) $f_n = \chi_{(n,n+1)}$
   (c) $f_n = n\chi_{[0,\frac{1}{n}]}$
   (d) $f_n = \chi_{[j2^{-k},(j+1)2^{-k}]}$ where $0 \le j < 2^k$ and $n = j + 2^k$

   converges to the 0-function
   (a) uniformly on $\mathbb{R}$
   (b) point-wise on $\mathbb{R}$
   (c) almost everywhere on $\mathbb{R}$
   (d) in $L^1(\mathbb{R})$.

   Which of these modes of convergence implies which others?

5. Let $f \in L^p(\mathbb{R})$ for $p \in [1, \infty)$ and $\lambda > 0$. Prove the inequality
$$|\{x \in \mathbb{R} : |f(x)| > \lambda\}| \le (\frac{\|f\|_p}{\lambda})^p,$$
where $|A|$ denotes the (Lebesgue) measure of the set $A \subset \mathbb{R}$.

6. Let $f \in C[0,1]$. Show that
$$\|f\|_p \to \|f\|_\infty \text{ for } p \to \infty.$$

7. Consider the set of all rational numbers $p/q \in (0,1)$ with denominator $q \le n$; call them $r_{n1}, r_{n2}, \dots, r_{nK}$ (where K depends on n). Define a function g_n by
$$g_n(x) = \Sigma_{i=1}^K \phi_n(x - r_{ni}),$$
where $\phi_n(u) = 1 - e^n u$ for $|u| \le e^{-n}$, $\phi_n(u) = 0$ for $|u| > e^{-n}$. Sketch the graph of g_n. Show that $g_n \in C([0,1])$, $\int_0^1 |g_n|^2\, dx \to 0$ as $n \to \infty$, and and $g_n(x) \to \chi_{\mathbb{Q}}(x)$ for rational x.

8. Show that if $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence and has a convergent subsequence then $(x_n)_{n=1}^{\infty}$ is convergent.

9. Assume that $(x_n)_{n=1}^{\infty}$ is a sequence in a Banach space such that for any $\epsilon > 0$ there is a convergent sequence $(y_n)_{n=1}^{\infty}$ such that $\|y_n - x_n\| < \epsilon$ for all n. Prove that $(x_n)_{n=1}^{\infty}$ is convergent. Give an example to show that the statement becomes false if Banach space is replaced by normed space.

10. Let l_c^{∞} denote the vector space with all convergent sequences $(x_n)_{n=1}^{\infty}$ of complex numbers equipped with the norm

$$\|(x_n)_{n=1}^{\infty}\|_{l_c^{\infty}} = \sup_n |x_n|.$$

Show that the space l_c^{∞} is complete.

11. Consider the vector space l^1 and set $\|\mathbb{x}\|_* = 2|\Sigma_{n=1}^{\infty} x_n| + \Sigma_{n=2}^{\infty}(1 + \frac{1}{n})|x_n|$ for $\mathbb{x} = (x_1, x_2, \ldots, x_n, \ldots) \in l^1$. Show that $\|\mathbb{x}\|_*$ defines a norm on l^1 and that the vector space l^1 is a Banach space with this norm. Is this norm equivalent to the standard norm $\|\mathbb{x}\|_{l^1}$?

12. Define $C_2^1([0,1])$ to be the space of continuously differentiable functions on $[0,1]$, with norm $\|f\| = (\int_0^1 (|f|^2 + |f'|^2)\, dx)^{1/2}$. Show that this is a proper definition of norm. Is this normed space complete?

13. What conditions must the function r satisfy in order that

$$\|f\| = \sup\{|f(x)r(x)| : 0 \leq x \leq 1\}$$

should define a norm on the vector space $C([0,1])$?

14. Let $BC([0,\infty))$ be the set of functions continuous for $x \geq 0$ and bounded. Show that for each $a > 0$, $\|f\|_a = (\int_0^{\infty} e^{-ax}|f(x)|^2\, dx)^{1/2}$ defines a norm on $BC([0,\infty))$, and $\|\cdot\|_a$ is not equivalent to $\|\cdot\|_b$ if $0 < b < a$. What about the case $a = 0$?

15. Show that every finite-dimensional normed space is complete.

16. Set $f_k(x) = \dfrac{\sin kx}{k^2}$, $0 \leq x \leq 1$, $k \in \mathbb{Z}_+$. Prove that the series $\Sigma_{k=1}^{\infty} f_k$ converges in $C([0,1])$.

17. Set for any $n \in \mathbb{Z}_+$, $f_n(x) = x^n - x^{n+1}$ and $g_n(x) = x^n - x^{2n}$ if $0 \leq x \leq 1$. Is any of the sequences $(f_n)_{n=1}^{\infty}$ and $(g_n)_{n=1}^{\infty}$ convergent in $C([0,1])$?

18. Let $M = \{x \in C([0,1]) : x(2^{-n}) = 0 \ \text{ all } \ n \in \mathbb{Z}_+\}$. Prove that M is a closed subset of $C([0,1])$.

19. Let $M = \{(x_n)_{n=1}^{\infty} \in c_0 : \Sigma_{n=1}^{\infty} 2^{-n} x_n = 0\} \subset c_0 \equiv \{(x_n)_{n=1}^{\infty} \in l^{\infty} : \lim_{n\to\infty} x_n = 0\}$. Show that M is a closed subspace in c_0.

20. Let E denote a normed space of finite dimension and let $e_1, \ldots, e_n$ be a basis of E. Set

$$f(x) = \Sigma_{k=1}^{n} x_k e_k, \ x = (x_1, \ldots, x_k) \in \mathbb{R}^n.$$

Show that f is continuous. Conclude from this that any two norms on E are equivalent.

21. Let E be a normed space and assume that $E \neq \{0\}$. Prove that there do not exist bounded linear operators A and B on E such that $AB - BA = I$.

22. Set $(Ax)(t) = x'(t)$ and $(Bx)(t) = tx(t)$, $0 < t < 1$, for $x \in C^\infty(]0,1[)$. Prove that $AB - BA = I$. Is it possible to find a norm on $C^\infty(]0,1[)$ such that A and B are bounded operators with respect to this norm[15]?

23. Let E and F be normed spaces and $T : E \to F$ a continuous mapping. Show that the $T(A)$ is compact in F if A is a compact set in E.

24. Let $T : E \to \mathbb{R}$ be a continuous mapping from a normed space E. Moreover let A be a compact set in E. Show that T attains its maximum and minimum on A.

25. Let $A : X \to X$ be a continuous mapping and assume $Ax \neq 0$ for all $x \in X$. Show that the mapping $B : x \mapsto Ax/\|Ax\|$ is continuous on X.

26. Find the norm of the linear functional

$$(x, y) \mapsto x - 7y$$

on $\mathbb{R}^2$ with respect to the norms l^p for $p = 1, 2$ and ∞.

27. For what values of the constant a does

$$u \mapsto \int_0^1 x^a u(x)\, dx$$

define a mapping $C([0,1]) \to \mathbb{C}$? For what values of a does it define a mapping $L^2([0,1]) \to \mathbb{C}$?

28. Show that the equation

$$\begin{cases} (Af)(x) = \int_{-\infty}^{+\infty} f(y)e^{-|x-y|}\, dy, & x \in \mathbb{R} \\ f \in L^2(\mathbb{R}) \end{cases}$$

defines a bounded linear operator A on $L^2(\mathbb{R})$.

29. Prove that any linear mapping from a finite-dimensional vector space into an arbitrary vector space must be continuous.

30. Let E be a normed space and L a linear functional on E. Furthermore, suppose there is a unit vector $x_0 \in E$ such that $\|x_0 - y\| \geq 1$ for every $y \in \mathcal{N}(L)$. Prove that $|Lx_0| = \|L\|$.

31. Find all linear mappings of $\mathbb{C}^n$ into $\mathbb{C}^m$ for $n, m \in \mathbb{Z}_+$.

32. Let A, B be two linear operators defined on a vector space E. Show that E must be infinite-dimensional if

$$AB = I \neq BA,$$

where I denotes the identity mapping on E. Give an example of such operators A and B on a vector space E.

[15] Hint: Show that $A^nB - BA^n = nA^{n-1}$ for $n = 1, 2, \ldots$

33. Let E be a vector space and $f : E \to \mathbb{R}$ a linear mapping. Suppose $x_0 \in E$ and $f(x_0) \neq 0$. Prove that any $x \in E$ may be written as $x = y + \alpha x_0$, where α is a scalar and $y \in \mathcal{N}(f)$. Show that this representation is unique.

34. Let f and g be two functionals on a vector space such that $\mathcal{N}(g) \subset \mathcal{N}(f)$. Prove that $f = \alpha g$, where α is a scalar.

35. Show that for any linear operator A on a n-dimensional vector space E, there are scalars $\alpha_0, \ldots, \alpha_{n^2}$, not all of them zero, such that

$$\Sigma_{k=0}^{n^2} \alpha_k A^k$$

is the zero operator.

36. Let B and C be linear operators on a finite-dimensional vector space E and suppose $\mathcal{N}(B) \subset \mathcal{N}(C)$. Show that there is a linear operator A on E such that $C = AB$.

37. Let E be a vector space of finite dimension and suppose $A : E \to E$ is a linear operator. Prove that $\mathcal{N}(A) = \{0\}$ if and only if $\mathcal{R}(A) = E$. Show that this is not true for vector spaces of infinite dimension.

38. Let E be a real normed space and let $T : E \to \mathbb{R}$ be a linear functional. Assume that $\mathcal{N}(T) \neq E$. Show that for all $x \in E$

$$\inf_{y \in \mathcal{N}(T)} \|x - y\| = \frac{|Tx|}{\|T\|}.$$

39. Show that the operator T on $C([0,1])$, where

$$(Tf)(t) = tf(t), \quad t \in [0,1],$$

is a bounded linear operator on $C([0,1])$.

40. Let A_n, A, B_n, B be bounded linear operators on a Banach space X. Show that $A_n \to A$ and $B_n \to B$ in $\mathcal{B}(X,X)$ implies $A_n B_n \to AB$ in $\mathcal{B}(X,X)$.

41. Let $A : X \to X$ be a bounded linear operator on a Banach space X. Show that $\Sigma_{n=0}^{\infty} \frac{1}{n!} A^n$ converges in $\mathcal{B}(X,X)$. Denote its sum by e^A. Show that for any integer $n > 0$, $(e^A)^n = e^{nA}$. Show that $e^O = I$ where O is the zero operator. Show that e^A is always invertible (even if A is not) and its inverse operator is e^{-A}. Show that if $AB = BA$, then $e^{A+B} = e^A e^B$. Show that $e^{A+B} = e^A e^B$ is not true in general.

42. Let A, B be invertible bounded linear operators on a Banach space X with $\|B^{-1}\|\|A - B\| < 1$. Show that if

$$\begin{cases} Ax = b \\ By = b \end{cases}$$

then

$$\|x - y\| \leq \frac{\|B^{-1}\|\|A - B\|}{1 - \|B^{-1}\|\|A - B\|}\|y\|.$$

Moreover also show that

$$\|x - y\| \leq \frac{\|B^{-1}\|^2\|A - B\|}{1 - \|B^{-1}\|\|A - B\|}\|b\|.$$

43. Let T be a bounded linear operator from a normed space E onto a normed space F. Assume that there is a constant $C > 0$ such that

$$\|Tx\| \geq C\|x\|$$

for all $x \in E$. Show that the inverse operator $T^{-1} : F \to E$ exists as a mapping and is a bounded linear operator.

44. Let $T : C([0,1]) \to C([0,1])$ be defined by

$$(Tf)(t) = \int_0^t f(s)\, ds.$$

Find $\mathcal{R}(T)$ and $T^{-1} : \mathcal{R}(T) \to C([0,1])$ satisfying $T^{-1}T = I_{C([0,1])}$. Is T^{-1} linear and bounded?

45. The operator $A : C([0,1]) \to C([0,1])$ is defined by the equation

$$(Af)(t) = f(t) + \int_0^t f(s)\, ds \quad 0 \leq t \leq 1.$$

Prove that $\mathcal{N}(A) = \{0\}$ and $\mathcal{R}(A) = C([0,1])$. Finally determine the inverse A^{-1} of A and show that A^{-1} is a bounded operator.

46. Let A be an $r \times n$-matrix with real entries. Consider A as a linear mapping from $\mathbb{R}^n$ into $\mathbb{R}^r$. Calculate or give an upper bound for the operator norm of A in
    (a) $\mathcal{B}(l^1, l^1)$
    (b) $\mathcal{B}(l^\infty, l^\infty)$

47. Let F be a subspace of a vector space E and let f be a functional on E such that $f(F)$ is not the whole scalar field of E. Show that $f(x) = 0$ for all $x \in F$.

48. Let $k \in \mathbb{Z}_+$ and set $L_k(f) = \int_0^\pi f(t) \sin kt\, dt$ for all $f \in C([0,\pi])$. Prove that $\|L_k\| = 2$ for all k.

49. Let

$$Lf = \int_0^{1/2} f(t)\, dt - \int_{1/2}^1 f(t)\, dt, \; f \in C([0,1]).$$

Prove that $\|L\| = 1$. Prove that there does not exist any $f \in C([0,1])$ such that $\|f\| = 1$ and $|Lf| = 1$.

50. Let E be a normed space and $A : E \to \mathbb{C}$ a bounded linear functional. Suppose there exists a vector $x_0 \in E$ such that $\|x_0\| = 1$ and $\|x_0 - x\| \geq 1$ for all $x \in \mathcal{N}(A)$. Show that $|Ax_0| = \|A\|$. Moreover, let $F = \{x \in C([0,1]) : \int_0^{1/2} x(t)\, dt = \int_{1/2}^1 x(t)\, dt\}$. Show that if $x_0 \in C([0,1])$ and $\|x - x_0\| \geq 1$ for all $x \in F$ then $\|x_0\| > 1$.

51. Show that

$$L(f) = \int_0^1 \frac{1}{\sqrt{x}} f(x)\, dx$$

defines a bounded linear functional on $C([0,1])$.

52. Show that
$$L(f) = \int_0^1 \frac{1}{\sqrt[3]{x}} f(x)\,dx$$
defines a bounded linear functional on $L^2([0,1])$.

53. Let T be defined by $T(\mathbb{x}) = (x_2, x_3, \ldots, x_{n+1}, \ldots)$ for all $\mathbb{x} = (x_1, x_2, \ldots, x_n, \ldots) \in l^2$. Show that $T \in \mathcal{B}(l^2, l^2)$ and calculate $\|T\|$.

54. Let $f(x)$ be a complex-valued function on $\mathbb{R}_+ = \{x : x > 0\}$ and let Lf be the function defined on $\mathbb{R}_+$ by
$$Lf(x) = \int_0^\infty f(y) e^{-xy}\,dy.$$
Show that L is a bounded[16] linear mapping $L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ with $\|L\| \leq \sqrt{\pi}$. Show that L is not a bounded[17] linear mapping $L^p(\mathbb{R}_+) \to L^p(\mathbb{R}_+)$ for $p \neq 2$.

55. (Non-orthogonal projections) A bounded linear operator P on a Banach space X will be called a **projector**[18] if $P^2 = P$.

    (a) Show that $I - P$ is a projector if P is. Show that if $x \in \mathcal{R}(P)$ then $Px = x$, and if $x \in \mathcal{R}(I - P)$ then $Px = 0$.

    (b) Show that for any projector P on a Banach space X, the range $\mathcal{R}(P)$ of P is a closed subspace, and is therefore itself a Banach space.

    (c) Show that any $x \in X$ can be uniquely expressed in the form $x = u + v$ with $u \in \mathcal{R}(P)$ and $v \in \mathcal{R}(I - P)$.

56. Let T be a linear mapping from a normed space V into a normed space W. Show that the range $\mathcal{R}(T)$ is a subspace of W. Show that the null-space (or kernel) $\mathcal{N}(T)$ is a subspace of V. If T is bounded, is it true that $T(V)$ and/or $\mathcal{N}(T)$ is closed?

57. Show that if $(x_1^{(n)}, x_2^{(n)}, \ldots) \to (x_1, x_2, \ldots)$ in l^p, then $x_k^{(n)} \to x_k$ in $\mathbb{R}$ for all $k \in \mathbb{N}$. If $x_k^{(n)} \to x_k$ in $\mathbb{R}$ for all $k \in \mathbb{N}$, is it true that $(x_1^{(n)}, x_2^{(n)}, \ldots) \to (x_1, x_2, \ldots)$ in l^p?

58. Let T be the linear mapping from $C^\infty(\mathbb{R})$ into itself given by $Tf = f'$. Show that T is surjective. Is T injective?

59. Consider the mapping T from $C[0,1]$ into itself, given by
$$Tf(t) = \int_0^t f(s)\,ds.$$
We assume that $C[0,1]$ is equipped with the sup-norm. Show that T is bounded and find $\|T\|$. Show that T is injective and find $T^{-1} : T(C[0,1]) \to C[0,1]$. Is T^{-1} bounded?

[16] Hint: write $f(y)e^{-yx} = (f(y)e^{-yx/2}y^{1/4})(e^{-yx/2}y^{-1/4})$ and use Hölder's inequality.

[17] Hint: Try $f(x) = e^{-ax}$

[18] Compare projections that are self-adjoint and satisfies $P^2 = P$. By projection we mean orthogonal projection.

60. Let T be a linear operator $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ satisfying that $f \geq 0$ implies that $Tf \geq 0$. Show that

$$\|T(|f|)\| \geq \|Tf\|$$

for all $f \in L^2(\mathbb{R})$. Show that T is bounded.

61. Define, for $h \in \mathbb{R}$, the operator τ_h on $L^2(\mathbb{R})$ by

$$\tau_h f(x) = f(x - h).$$

Show that τ_h is bounded.

62. Let V be a Banach space and let $T \in \mathcal{B}(V, V)$ such that T^{-1} exists and belongs to $\mathcal{B}(V, V)$. Show that if $\|T\| \leq 1$ and $\|T^{-1}\| \leq 1$, then

$$\|T\| = \|T^{-1}\| = 1,$$

and $\|Tf\| = \|f\|$ for all $f \in V$.

63. Consider the operator

$$Af(x) = \frac{1}{\sqrt{\pi}} \int_0^x \frac{f(t)}{\sqrt{x-t}}\, dt, \quad x \in [0, 1]$$

whenever this expression makes sense. Show that $Af \in L^\infty[0, 1]$ if $f \in L^p[0, 1]$, $p > 2$. Find the operator $B = A^2$, i.e. find the kernel $k(x, t)$ such that

$$Bf(x) = \int_0^x k(x, t) f(t)\, dt$$

for $f \in L^p[0, 1]$, $p > 2$. Show that $B : L^p[0, 1] \to L^\infty[0, 1]$, $1 \leq p \leq \infty$ is bounded. Solve the equation

$$(I - A)f(x) = 1$$

formally by a Neumann series, and express f as

$$f(x) = g(x) + Ah(x)$$

where g and h are known functions. Insert and show that this formal solution is a solution.

6.4 Fixed point techniques

Key words: contractions, Banach's fixed point theorem, Brouwer's fixed point theorem, Schauder's fixed point theorem

1. Show that the Banach fixed point theorem is valid for metric spaces (X, d) as follows: Let (X, d) be a complete[19] metric space and let F be a closed set in X. Assume that $T : F \to F$ is a contraction mapping on F. Then T has a unique fixed point.

2. Consider the metric space (X, d), where $X = [1, \infty)$ and d the usual distance. Let $T : X \to X$ be given by
$$T(x) = \frac{x}{2} + \frac{1}{x}.$$
Show that T is a contraction and find the minimal contraction constant. Find also the fixed point.

3. Let T be a mapping from a metric space (X, d) into itself such that
$$d(T(x), T(y)) < d(x, y)$$
for all $x, y \in X$, $x \neq y$. Show that T has at most one fixed point. Show[20] that T not necessarily have a fixed point.

4. A mapping $T : \mathbb{R} \to \mathbb{R}$ satisfies a Lipschitz-condition with constant k if
$$|T(x) - T(y)| \leq k|x - y|$$
for all $x, y \in \mathbb{R}$.
   (a) Is T a contraction?
   (b) If T is a C^1–function with bounded derivative, show that T satisfies a Lipschitz-condition.
   (c) If T satisfies a Lipschitz-condition, is T then a C^1-function with bounded derivative?
   (d) Assume that $|T(x) - T(y)| \leq k|x - y|^\alpha$ for some $\alpha > 1$. Show that T is a constant.

5. Let X be a Banach space and let T, S be two mappings from X into X (not necessarily linear). Assume that $TS = ST$ and that T has a unique fixed point. Show that S has a fixed point. What can be said if T has more than one fixed point?

6. Let F be a compact set in a normed space X and let $T : F \to F$ have the property
$$\|T(x) - T(y)\| < \|x - y\|, \quad \text{all } x \neq y \in F.$$
Show that T has a unique fixed point.

[19] see footnote to Baire's theorem below

[20] Hint: e.g. consider $T(x) = x + \frac{1}{x}$ for $x \in [1, \infty)$.

7. Let X be a Banach space and T a mapping on X satisfying

$$\|T(x) - T(y)\| \geq K\|x - y\| \text{ all } x, y \in X,$$

where $K > 1$. Assume that $T(X) = X$. Show that T has a unique fixed point.

8. We consider the vector space $\mathbb{R}^n$ with l^1-norm and a mapping $T : \mathbb{R}^n \to \mathbb{R}^n$ given by $Tx = Cx + b$, where $C = (c_{ij})$ is an $n \times n$-matrix and $b \in \mathbb{R}^n$. Show that T is a contraction if

$$\Sigma_{i=1}^n |c_{ij}| < 1 \text{ for all } j = 1, 2, \ldots, n.$$

If we instead use the l^2-norm, show that T is a contraction if

$$\Sigma_{i=1}^n \Sigma_{j=1}^n |c_{ij}|^2 < 1.$$

9. Use Banach fixed point theorem to find a root (given to four decimal places) of the equation

$$x^2 - \sin^2 x - 1 = 0$$

in the interval $[1, \sqrt{2}]$.

10. Suppose $0 < L < \sqrt{(\sqrt{5} - 1)/2}$. Show that there exists a unique $u \in C([0, 1])$ such that

$$u(x) = \int_0^L \sqrt{1 + (x - y)^2} \cos(u(y))\, dy + \sin(e^{-x}), \quad 0 \leq x \leq L.$$

11. Show that the equation

$$u(x) = \int_0^p \sqrt{1 + (x - y)^2} \cos u(y)\, dy + \sin(\pi e^{-4x^2})$$

has a unique solution in $C([0, p])$ for $p > 0$ small enough. Give an upper estimate on p?

12. Suppose $\lambda \in \mathbb{C}$. Solve the equation

$$\begin{cases} u(x) - \lambda \int_0^1 xyu(y)\, dy = f(x) & 0 \leq x \leq 1 \\ u \in C([0, 1]) \end{cases}$$

where $f \in C([0, 1])$ is a given function.

13. Suppose $\lambda \in \mathbb{C}$. Solve the equation

$$\begin{cases} u(x) - \lambda \int_0^x xyu(y)\, dy = f(x) & 0 \leq x \leq 1 \\ u \in C([0, 1]) \end{cases}$$

where $f \in C([0, 1])$ is a given function.

14. Suppose $f \in C([0, 1])$. Prove that the following equation possesses a unique solution where

$$\begin{cases} u(x) - 5 \int_0^{1-x} u(y) \min(x, y)\, dy = f(x) & 0 \leq x \leq 1 \\ u \in C([0, 1]). \end{cases}$$

15. Let P be the set of all ordered pairs $f = (f_1, f_2)$ of real-valued continuous functions on $[0,1]$. Show that P is a Banach space if we define addition and scalar multiplication in the obvious way, and define $\|f\|_P = \max\{\|f_1\|_\infty, \|f_2\|_\infty\}$. Show that the coupled integral equations

$$\begin{cases} u(x) = \lambda \int_0^1 e^{xy} \frac{u(y)}{1+u^2(y)+v^2(y)}\, dy \\ \\ v(x) = \mu \int_0^1 e^{xy} \frac{u(y)v(y)}{1+u^2(y)+v^2(y)}\, dy \end{cases}$$

have no nontrivial solutions if $|\lambda| < 1/2e$ and $|\mu| < 1/e$.

16. Consider the equation[21]

$$3u(x) = x + (u(x))^2 + \int_0^1 |x - u(y)|^{1/2}\, dy.$$

Show that it has a continuous solution u satisfying $0 \le u(x) \le 1$ for $0 \le x \le 1$.

17. Let S be the set $\{f \in C([0,1]) : \|f\|_\infty \le 1,\ f(0) = 0,\ f(1) = 1\}$ and the operator $T : S \to S$ defined by $(Tf)(x) = f(x^2)$. Show that S is a closed bounded convex set and that T is a continuous operator with no fixed point.

18. Let c_0 denote the vector space

$$c_0 = \{(x_n)_{n=1}^\infty \in l^\infty : \lim_{n\to\infty} x_n = 0\}$$

with the norm

$$\|(x_n)_{n=1}^\infty\|_{c_0} = \max_n |x_n|.$$

Define $T : c_0 \to c_0$ by $T((x_n)_{n=1}^\infty) = (z_n)_{n=1}^\infty$, where

$$\begin{cases} z_1 = \frac{1}{2}(1 + \|(x_n)_{n=1}^\infty\|) \\ z_n = (1 - 2^{-n})x_{n-1},\ n \ge 2 \end{cases}$$

Show that T maps the closed unit ball in c_0 into itself and that

$$\|T(x) - T(y)\| < \|x - y\|$$

for all x, y, $x \ne y$, in the unit ball in c_0. Moreover, show that T have no fixed points in the unit ball in c_0.

19. Let T denote the mapping $(x,y) \mapsto (x+y, y-(x+y)^3)$ on $\mathbb{R}^2$. Show that T is an odd mapping, i.e. $T(-x,-y) = -T(x,y)$, and that $(0,0)$ is the only fixed point of T. Moreover show that $(2,-4)$ and $(-2,4)$ are fixed points of T^2. Can T be a contraction?

20. T denote the mapping $(x,y) \mapsto (y^{1/3}, x^{1/3})$ on $\mathbb{R}^2$. What are the fixed points of T? What happens when you iterate, starting from various places in $\mathbb{R}^2$ (find out by numerical experiments)? In what regions is T a contraction?

[21] Hint: Krasnoselskii's fixed point theorem

21. Let T be a contraction on a Banach space E, i.e.

$$\|Tx - Ty\| \le \alpha \|x - y\|$$

for all $x, y \in E$ for some $\alpha \in (0,1)$, and assume that S is a mapping on E such that $\|Tx - Sx\| \le \lambda$ for all $x \in E$ for some constant $\lambda > 0$. Show that

$$\|T^n x - S^n x\| \le \lambda \frac{1-\alpha^n}{1-\alpha}$$

for $n \in \mathbb{Z}_+$. Show that if S has a fixed point y then

$$\|x - y\| \le \lambda \frac{1}{1-\alpha},$$

where x is the unique fixed point for T. Finally show that if $y_n = S^n y_0$ then

$$\|x - y_n\| \le \frac{1}{1-\alpha}(\lambda + \alpha^n \|y_0 - Sy_0\|),$$

provided x is the fixed point for T. What is the significance of this formula in applications?

22. Consider the equation

$$x(t) - \mu \int_0^1 k(t,s)x(s)\,ds = v(t), \quad t \in [0,1], \tag{21}$$

where $k \in C([0,1] \times [0,1])$ and $v \in C([0,1])$. Moreover assume that

$$\max_{(t,s)\in[0,1]\times[0,1]} |k(t,s)| \le c.$$

Show that (21) has a unique solution $x \in C([0,1])$ provided $|\mu|c < 1$ using the iterative sequence

$$x_{n+1}(t) = v(t) + \mu \int_0^1 k(t,s)x_n(s)\,ds. \tag{22}$$

Next set

$$Sx(t) = \int_0^1 k(t,s)x(s)\,ds$$

and

$$z_{n+1} = \mu S z_n.$$

Choosing $x_0 = v$ show that (22) yields the so called **Neumann series**

$$x = \lim_{n\to\infty} x_n = v + \mu S v + \mu^2 S^2 v + \mu^3 S^3 v + \dots$$

Show that in the Neumann series we can write

$$S^n v(t) = \int_0^1 k_{(n)}(t,s)v(s)\,ds, \quad n = 1, 2, 3, \dots$$

where the so called iterated kernel $k_{(n)}$ is given by

$$k_{(n)}(t,s) = \int_0^1 \cdots \int_0^1 k(t,t_1)k(t_1,t_2)\cdots k(t_{n-1},s)\,dt_1\cdots dt_{n-1}.$$

Show that the solution of (21) can be written

$$x(t) = v(t) + \mu \int_0^1 \tilde{k}(t,s,\mu)v(s)\,ds$$

where

$$\tilde{k}(t,s,\mu) = \Sigma_{j=0}^{\infty}\mu^j k_{(j+1)}(t,s).$$

23. Use the methods in the above problem to solve

$$x(t) - \mu \int_0^1 cx(s)\,ds = v(t), \quad t \in [0,1]$$

where c is a constant.

24. (a) A nonlinear version of the Volterra operator is defined as follows: $(Lu)(x) = \int_0^x K(x,y)f(y,u(y))\,dy$ where K and f are continuous functions, and $|f(y,u) - f(x,v)| \leq N|u-v|$ for all u,v,x,y where N is a constant. Then L maps $C([0,T])$ into itself for any $T > 0$. Give an example to show that L is not a contraction on $C([0,T])$ with the usual norm $\|u\| = \sup|u(x)|$.

    (b) Show that for any $a > 0$, $\|u\|_a = \sup\{e^{-ax}|u(x)| : 0 \leq x \leq T\}$ defines a norm on $C([0,T])$ which is equivalent to the usual norm. Deduce that $C([0,T])$ with the norm $\|\cdot\|_a$ is a Banach space.

    (c) Set $M = \max\{|K(x,y)| : 0 \leq x,y \leq T\}$. Show that $\|Lu - Lv\|_a \leq MN/a(1 - e^{-aT})\|u-v\|_a$ for all $u,v \in C([0,T])$. Deduce that for any $T > 0$ the integral equation $u = Lu + g$, where g is a given continuous function, has a unique solution.

25. Let $f : \mathbb{R} \to \mathbb{R}$ be a C^1-mapping and assume that $|f'(x)| \leq c < 1$ for all $x \in \mathbb{R}$. Show that $g : \mathbb{R} \to \mathbb{R}$ is surjective , where $g(x) = x + f(x)$.

26. Let X and Y be Banach spaces and let $T : X \to Y$ be a mapping having the following property: There exists a number $C > 0$ such that for any $x,y \in X$ we have

$$|T(x+y) - T(x) - T(y)| \leq C.$$

    (a) Show that there exists a unique additive[22]mapping[23] $S : X \to Y$ such that $T - S$ is bounded in the sup-norm.

    (b) If T is continuous, prove that S is continuous and linear.

[22] S additive means that

$$S(x+y) = S(x) + S(y)$$

for all $x,y \in X$.

[23] Hint: Show that $S(x) = \lim_{n\to\infty} \frac{1}{2^n}T(2^n x)$ does the job.

27. (Newton's iteration) Let f be a real C^2-function on an interval $[a, b]$, and let $\xi \in (a, b)$ be a simple zero of f. Show that Newton's method

$$x_{n+1} = T(x_n) \equiv x_n - \frac{f(x_n)}{f'(x_n)}$$

is a contraction in some neighborhood of ξ.

28. (Halley's iteration) In 1694 Edmund Halley, well-known for first computing the orbit of the Halley comet, presented the following algorithm for computing roots of a polynomial. Show that if f is a real C^3-function on an interval $[a, b]$, and if $\xi \in (a, b)$ is a simple zero of f then the algorithm

$$x_{n+1} = T(x_n) \equiv x_n - \frac{f(x_n)}{f'(x_n) - \frac{f''(x_n)f(x_n)}{f'(x_n)}}$$

is a contraction in some neighborhood of ξ.

29. For each of the following sets give an example of a continuous mapping of the set into itself that has no fixed points:
    (a) the real line $\mathbb{R}$
    (b) the interval $(0, 1]$
    (c) the set $[0, 1] \bigcup [2, 3]$

30. Give an example of a mapping of the closed interval $[0, 1]$ into itself that has no fixed points (and hence is not continuous).

31. Let $f : S^1 \to \mathbb{R}$ be a continuous function, where S^1 denotes the unit circle centered at the origin. Show that there is an $x \in S^1$ such that $f(x) = f(-x)$. This result is called the **Borsuk-Ulam theorem** for the circle.

32. Let A and B be two bounded plane figures. Show that there is a line dividing each into two parts of equal area.

33. Let K be a closed disc in the plane $\mathbb{R}^2$ and let C be its boundary circle. Assume that the function f is a continuous mapping $K \to \mathbb{R}^2$ such that $f|_C = I$ and that g is a continuous mapping $K \to K$. Show that there is a point $p \in K$ such that $f(p) = g(p)$.

34. Prove **Baire's theorem** [Let X be a complete[24] metric space.
    (a) If $\{U_n\}_{n=1}^{\infty}$ is a sequence of open dense subsets of X, then $\bigcap_{n=1}^{\infty} U_n$ is dense in X.
    (b) X is not a countable union of nowhere dense sets.]

[24] For the definition of a metric space X with metric d see exercise 6 in the section "normed spaces". We say that a set $A \subset X$ is open if for each $x \in A$ there is an $r > 0$ such that $\{y \in X : d(x, y) < r\} \subset A$. A set $B \subset X$ is closed if its complement B^c is an open set. Given a subset E of X. The intersection of all closed sets in X containing E is a closed set, is called the closure of E and is denoted $\overline{E}$. The union of all open sets in X contained in E is an open set, is called the interior of E and is denoted by E^0. We say that a set E in X is dense in X if $\overline{E} = X$ and we say that E is nowhere dense if $(\overline{E})^0 = \emptyset$. Finally, a metric space is called complete if for each sequence $\{x_n\} \subset X$ such that $d(x_n, x_m) \to 0$ as $n, m \to \infty$ there exists an x such that $d(x_n, x) \to 0$ as $n \to \infty$.

35. Use Baire's theorem to show the existence of $f \in C([0,1])$ that is nowhere differentiable. [Hint: Consider the sets E_n of all $f \in C([0,1])$ for which there exists $x_0 \in [0,1]$ (depending on f) such that

$$|f(x) - f(x_0)| \leq n|x - x_0|$$

for all $x \in [0,1]$. Show that E_n is nowhere dense in $C([0,1])$.]

36. Prove **Banach-Steinhaus theorem** [Suppose X is a Banach space and Y is a normed space and that $\mathcal{A} \subset \mathcal{B}(X,Y)$. Moreover assume that

$$\sup_{T \in \mathcal{A}} \|Tx\| < \infty \quad \text{for all} \quad x \in X.$$

Then

$$\sup_{T \in \mathcal{A}} \|T\| < \infty.]$$

37. Use Banach-Steinhaus theorem[25] to show the existence of a continuous function on $[-\pi, \pi]$ such that its Fourier series diverges at 0.

38. Prove **Perron's theorem**, i.e. prove that an $n \times n$-matrix, whose elements are all positive, has at least one positive eigenvalue and that the elements of the corresponding eigenvector are all positive.

39. A linear integral operator with a positive kernel is a natural analogue of the positive matrix in Perron's theorem. Use Schauder's theorem to prove that an integral operator with positive continuous kernel has a positive eigenvalue.

40. Let $T : \bar{B}(0,1) \to \bar{B}(0,1)$ where $\bar{B}(0,1)$ is the closed unit ball in $\mathbb{R}^n$ centered at the origin. Assume that

$$|T(x) - T(y)| \leq |x - y|$$

for all $x, y \in \bar{B}(0,1)$ where $|\cdot|$ denotes the Euclidean distance. Show that T has a fixed point using

(a) Brouwder's fixed point theorem

(b) Banach's contraction theorem[26]

41. Prove **Arzela-Ascoli's theorem** [Let $A \subset C([0,1])$. It follows that that $\overline{A}$ is compact if and only if

(a) (uniform boundedness) there exists an $M < \infty$ such that

$$\sup_{x \in [0,1],\, f \in A} |f(x)| \leq M$$

and

[25] Hint: Let $T_n f$ denote the n–th partial sum of the Fourier series of f.

[26] Hint: Consider $T_n = (1 - \frac{1}{n})T$.

(b) (equicontinuity) for all $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon$$

for all $x, y \in [0,1]$ with $|x - y| < \delta$ and all $f \in A$.]

42. Let M be a bounded set in $C([0,1]$,not necessarily compact. Show that the set of all functions $F(x) = \int_0^x f(t)\,dt$ with $f \in M$ is relatively compact.

43. Prove **Sperner's lemma** [Let Δ be a closed triangle with vertices v_1, v_2, v_3 and let τ be a triangulation of Δ. This means that $\tau = \{\Delta_i\}_{i\in I}$ where Δ_i are closed triangles with the properties

   (a) $\Delta = \bigcup_{i\in I} \Delta_i$

   (b) For every $i, j \in I$, $i \neq j$, we have

$$\Delta_i \bigcap \Delta_j = \begin{cases} \emptyset & \text{or} \\ \text{common vertex} & \text{or} \\ \text{common side} \end{cases}$$

   Moreover let $\mathcal{V}$ denote the set of all vertices of the triangles Δ_i and let $c : \mathcal{V} \to \{1,2,3\}$ be a function that satisfies the following conditions:

   (a) $c(v_i) = i$ for $i = 1,2,3$

   (b) $v \in \mathcal{V} \bigcap v_i v_j \in \{i,j\}$ for $i,j \in \{1,2,3\}$ where $v_i v_j$ denotes the line segment between v_i and v_j.

   Then there exists a triangle Δ_i such that the vertices of the triangle take different values.]

44. Prove Brouwer's fixed point theorem in a special case[27] (n=2): Let $T : K \to K$ be a continuous mapping where K denotes the set $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : \Sigma_{i=1}^3 x_i = 1,\ x_i \geq 0 \text{ all } i\}$. Then T has a fixed point.

45. Let $(a_n)_{n=1}^\infty$ be a bounded sequence, i.e. $(a_n)_{n=1}^\infty \in l^\infty$. Show, by using Banach's fixed point theorem[28], that there exists a bounded sequence $(x_n)_{n=1}^\infty$ that solves the equations

$$x_{n-1} + 4x_n + x_{n+1} = a_n, \quad n = 1, 2, \ldots,$$

   where $x_0 = 1$.

---

[27] Consider a sequence of finer and finer triangulations of K and make use of the function $c : K \to \{1,2,3\}$ defined by

$$c(\mathbb{x}) = \min\{i : (T(\mathbb{x}))_i < x_i\}$$

where $\mathbb{x} = (x_1, x_2, x_3)$. Note that the function c is well-defined provided T has no fixed point, and apply Sperner's lemma.

[28] Consider the mapping

$$x_n \mapsto \frac{1}{4}(a_n - x_{n-1} - x_{n+1}), \quad n = 1, 2 \ldots.$$

6.5 Hilbert spaces

Key words: inner product, inner product space, polarization identity, Hilbert space, orthogonality, strong/weak convergence, orthonormal sequence, Gram–Schmidt orthonormalization process, complete sequence, orthogonal complement, convex set, orthogonal projection and decomposition, separable Hilbert space

1. Let $z_1, \ldots, z_n$ be complex numbers. Show that
$$|z_1 + \ldots + z_n| \leq \sqrt{n}\|(z_1, \ldots, z_n)\|.$$

2. Let x, y be vectors in a complex vector space with inner product, and assume that
$$\|x+y\|^2 = \|x\|^2 + \|y\|^2.$$
Does this imply that $\langle x, y \rangle = 0$?

3. Let H be a Hilbert space. Show that
$$\|x - z\| = \|x - y\| + \|y - z\|$$
if and only if $y = \alpha x + (1-\alpha)z$ for some $\alpha \in [0, 1]$.

4. Let $\|\cdot\|$ denote the norm in a Hilbert space. Prove that
$$\|x+y\|\,\|x-y\| \leq \|x\|^2 + \|y\|^2$$
and
$$\|x+y\|^2 - \|x-y\|^2 \leq 4\|x\|\,\|y\|.$$

5. Let E be an inner product space. Show that for $x, y \in E$, $x \perp y$ if and only if $\|\alpha x + \beta y\|^2 = \|\alpha x\|^2 + \|\beta y\|^2$ for all scalars α and β.

6. Show that $C([0,1])$ (equipped with the sup-norm) is not an inner product space.

7. Prove that any complex Banach space with norm $\|\cdot\|$ satisfying the parallelogram law is a Hilbert space with the inner product
$$\langle x, y \rangle = \frac{1}{4}[\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2],$$
and $\|x\|^2 = \langle x, x \rangle$.

8. Let $x_1, x_2, \ldots, x_N$ be linearly independent vectors in an inner product space, with $N = \binom{n+1}{2}$. Show that there are orthonormal vectors $y_1, y_2, \ldots, y_n$ such that
$$y_i = \Sigma_{j \in A_i} \lambda_j x_j, \; i = 1, 2, \ldots, n,$$
where $A_1, A_2, \ldots, A_n$ are disjoint subsets of $\{1, 2, \ldots, N\}$.

9. Let $T : E \to E$ be a bounded linear operator on a complex inner product space. Show that $T = 0$ if $\langle Tx, x\rangle = 0$ for all $x \in E$. Show that this does not hold in the case of real inner product spaces.

10. Suppose $x_n \to x$ and $y_n \to y$ in a Hilbert space H and $\alpha_n \to \alpha$ in $\mathbb{C}$. Prove that

    (a) $x_n + y_n \to x + y$

    (b) $\alpha_n x_n \to \alpha x$

    (c) $\langle x_n, y_n\rangle \to \langle x, y\rangle$

    (d) $\|x_n\| \to \|x\|$

11. Suppose $x_n \overset{w}{\to} x$ and $y_n \overset{w}{\to} y$ in a Hilbert space H and $\alpha_n \to \alpha$ in $\mathbb{C}$. Prove or disprove that

    (a) $x_n + y_n \overset{w}{\to} x + y$

    (b) $\alpha_n x_n \overset{w}{\to} \alpha x$

    (c) $\langle x_n, y_n\rangle \to \langle x, y\rangle$

    (d) $\|x_n\| \to \|x\|$

12. Let $(e_n)_{n=1}^{\infty}$ be an ON-basis for H. Assume that the sequence $(f_n)_{n=1}^{\infty}$ in H satisfies the conditions $\|f_n\| = 1$ and $f_n \in \{e_1, e_2, \dots, e_n\}^{\perp}$ for $n = 1, 2, \dots$. Show that $f_n \overset{w}{\to} \mathbf{0}$.

13. Suppose $x_n \overset{w}{\to} x$ in a Hilbert space H. Show[29] that there is a positive constant M such that

$$\sup_n \|x_n\| \leq M.$$

14. Let $(x_n)_{n=1}^{\infty}$ be a bounded sequence, i.e. $\sup_n \|x_n\| \leq M$, in a separable Hilbert space H. Show that there is a subsequence $(x_{n_k})_{k=1}^{\infty}$ and an $x \in H$ such that

$$x_{n_k} \overset{w}{\to} x.$$

    What happens if H is not separable?

15. Suppose $x_n \overset{w}{\to} x$ in a Hilbert space H. Show that there exists a subsequence $(x_{n_k})_{k=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ such that

$$\frac{1}{m}\Sigma_{k=1}^{m} x_{n_k} \to x \ \text{ i } \ H,$$

    då $m \to \infty$.

16. Consider $\mathbb{R}^n$ as a Hilbert space with the standard inner product and the corresponding norm, i.e. the Euclidean metric. Assume that S is a closed convex set in $\mathbb{R}^n$ and that for each $x \in \mathbb{R}^n$ there exists a unique $y \in S$ such that

$$\|x - y\| = \sup_{z \in S} \|x - z\|.$$

    Show that S consists of a single element.

[29] Hint: Use Banach-Steinhaus theorem above

17. Let $(e_k)_{k=1}^n$ be a sequence of vectors in a Hilbert space H. Assume that $\|e_k\| = 1$ for all k. Show[30] that

$$\Sigma_{k=1}^n |\langle x, e_k\rangle|^2 \leq \|x\|^2 (1 + (\Sigma_{\substack{k,l=1\\k\neq l}}^n |\langle e_k, e_l\rangle|^2)^{\frac{1}{2}})$$

for all $x \in H$. Note that if $(e_k)_{k=1}^n$ is an ON-sequence in H then the statement is called Bessel's inequality.

18. Assume that M is a closed subspace of a Hilbert space H. Let $\{x_n\}_{n=1}^\infty$ be a sequence converging to x in H. Moreover let $x_n = y_n + z_n$, $n = 1, 2, \ldots$, be the orthogonal decomposition of x_n with $y_n \in M$ and $z_n \in M^\perp$. Show that y_n converges to y and z_n converges to z where $x = y + z$ is the orthogonal decomposition of x.

19. Consider the inner product space X of the vector space $C([0,1])$ with the inner product of $L^2([0,1])$. Set $S = \{f \in C([0,1]) : f(x) = 0 \text{ for } x \in [0, \frac{1}{2}]\}$. Show that S is a closed subspace of X and calculate $S^\perp$. Is $X = S + S^\perp$?

20. What is the orthogonal complement of all even functions in $L^2([-1,1])$?

21. Let M be the subset $\{(x_n)_{n=1}^\infty : x_{2n} = 0 \text{ for all } n \in \mathbb{Z}_+\}$ in l^2. Give $M^\perp$ and $M^{\perp\perp}$.

22. Let A be a subset of a Hilbert space. Show that

$$A^{\perp\perp} = \overline{\text{Span}A}.$$

23. Let A and B, $\emptyset \neq A \subset B$, be subsets of an inner product space. Show that

    (a) $B^\perp \subset A^\perp$

    (b) $A^{\perp\perp\perp} = A^\perp$.

24. Let $M \neq \emptyset$ be a subset of a Hilbert space H. Show that $\text{Span}\, M$ is dense in H if and only if $M^\perp = \{0\}$. By the span of a set $\mathcal{A}$ we mean all finite linear combinations of the elements in the set $\mathcal{A}$.

25. Let $(x_n)_{n=1}^\infty$ be a complete orthonormal sequence in a Hilbert space H. Show that

$$\langle x, y\rangle = \Sigma_{n=1}^\infty \langle x, x_n\rangle \overline{\langle y, x_n\rangle}$$

for all $x, y \in H$. Also show that the reverse implication is true.

26. Let $(x_n)_{n=1}^\infty$ be an orthonormal sequence in a Hilbert space H. Show that $(x_n)_{n=1}^\infty$ is complete if and only if the closure of the span of $(x_n)_{n=1}^\infty$ equals H.

27. If $(x_n)_{n=1}^\infty$ is a complete orthonormal set for a vector subspace S of a Hilbert space H, then any $x \in S$ can be expressed in the form $x = \Sigma c_n x_n$. Conversely, if $y = \Sigma c_n x_n$, does if follow that $y \in S$? What happens if S is a Hilbert subspace of H?

28. Given a convergent infinite series, one cannot in general rearrange the terms; if the sequence (v_n) is a rearrangement of a series (u_n), and $\Sigma u_n = U$, then Σv_n need not equal U, unless Σu_n converges absolutely. However, prove that if (e_n) is a complete orthonormal set and (f_n) is a sequence obtained by arranging (e_n) in a different order, then (f_n) is a complete orthonormal set, and therefore the series $x = \Sigma\langle x, e_n\rangle e_n$ can be rearranged.

[30]Hint: Note that $\Sigma|\langle x, e_k\rangle|^2 = \langle x, \Sigma\langle x, e_k\rangle e_k\rangle$.

29. (A space with no complete ON sequence) The set of all periodic functions $\mathbb{R} \to \mathbb{C}$ is clearly not a vector space. But if we consider the set M of functions which are sums and products of finitely many periodic functions, we obtain a vector space. The elements of M are called **almost-periodic functions**. It can be proved that for any $f, g \in M$,

$$\lim_{T\to\infty} \left(\frac{1}{2T}\int_{-T}^{T} f(t)\overline{g(t)}\, dt\right)$$

exists and defines an inner product on M. Verify that any two members of the family of functions e^{iat}, where a is real, are orthogonal in the inner product space M. Deduce that M has no countable basis.

30. Find an orthonormal basis of the subspace $\mathrm{Span}\{1+x, 1-x\}$ of $L^2([0,1])$.

31. Let P and Q denote orthogonal projections onto two subspaces in a Hilbert space. Prove that $\|P - Q\| \le 1$.

32. Suppose S is a closed convex subset of a Hilbert space H and let P_S denote the orthogonal projection onto S, i.e. for any $x \in H$, $P_S(x)$ denotes the point in S, which is nearest to x. Prove that
$$\|P_S(x) - P_S(y)\| \le \|x - y\| \text{ for all } x, y \in H.$$

33. In the vector space $\mathbb{R}^n$ use the norm $\|u\| = \Sigma|u_i|$. Let $x = (1, -1, 0, \ldots, 0)$ and let E be the subspace $\{(t, t, 0, \ldots, 0) : t \in \mathbb{R}\}$. Setting $y_t = (t, t, 0, \ldots, 0)$ for the elements of E, show that all y_t with $|t| \le 1$ have the same distance from x, and are closer to x than any y_t with $|t| > 1$. This shows that the best approximation in a subspace can be non-unique in normed spaces, though in Hilbert spaces they are unique. Deduce that the norm $\Sigma|u_i|$ cannot be obtained from any inner product.

34. Let $H = \{f \in L^2([0,1]) : f' \in L^2([0,1])\}$, and for $f, g \in H$ define
$$\langle f, g\rangle = f(0)g(0) + \int_0^1 f'(s)g'(s)\, ds.$$
Take L^2 here to be the space of real functions. Show that H is a Hilbert space. For each $t \in [0,1]$ define a function $R_t \in H$ by $R_t(s) = 1 + \min(s,t)$, where $\min(s,t)$ denotes the smaller of s and t. Show that $\langle f, R_t\rangle = f(t)$ for all $f \in H$.

    Now consider the following problem in approximation theory. The interval $[0,1]$ is divided into subintervals given by numbers $0 = t_1 < t_2 < \ldots < t_n = 1$. Given a function f, we wish to approximate it by a piecewise linear function F which is linear in each subinterval. Show that the set of all such functions F is the subspace spanned by $\{R_{t_i} : i = 1, 2, \ldots, n\}$. Show that the best piecewise linear approximation to f in the sense of the norm corresponding to the above inner product in H is the piecewise linear function F which equals f at the points t_i.

35. Suppose $A : H \to H$ is a linear mapping that satisfies
$$\langle Ax, y\rangle = \langle x, Ay\rangle \text{ all } x, y \in H.$$
Prove that A is a continuous mapping.[31]

[31] Hint: Apply Banach–Steinhaus theorem

36. Let $(x_n)_{n=1}^{\infty}$ be a complete ON-sequence in a Hilbert space H and let $(y_n)_{n=1}^{\infty}$ be another ON-sequence such that

$$\Sigma_{n=1}^{\infty}\|x_n - y_n\|^2 < 1.$$

Show that the ON-sequence $(y_n)_{n=1}^{\infty}$ also is complete.

37. Let $(u_n)_{n=1}^{\infty}$ be an orthonormal sequence in $L^2([0,1])$. Show that the sequence is an orthonormal basis if

$$\Sigma_{n=1}^{\infty}\Big|\int_0^x \overline{u_n(t)}\,dt\Big|^2 = x, \quad \text{for all } x \in [0,1].$$

6.6 Linear operators on Hilbert spaces

Key words: bilinear functional, quadratic form, coercive functional, adjoint operator, self-adjoint operator, inverse operator, normal operator, isometric operator, unitary operator, positive operator, projection operator, compact operator, finite-dimensional operator, eigenvalues/eigenvectors, resolvent, spectrum, unbounded operators

1. Let A be a self-adjoint operator on a Hilbert space H and assume that $\overline{\mathcal{R}(A)} = H$. Show that $A : H \to \mathcal{R}(A)$ is an invertible mapping.

2. Assume that $A_n \to A$ in $\mathcal{B}(H, H)$, where H is a Hilbert space. Show that A is self-adjoint if all A_n are self-adjoint.

3. Let A be a linear compact operator on a Hilbert space H. Prove that $I + A$ is a compact operator if and only if H is finite-dimensional.

4. Let B be a bounded linear operator on a Hilbert space. Prove that
$$\mathcal{R}(B)^{\perp} = \mathcal{N}(B^*)$$
and
$$\overline{\mathcal{R}(B)} = \mathcal{N}(B^*)^{\perp}.$$

5. Let A be a compact linear operator on a Hilbert space H. Prove that $\mathcal{R}(I - A)$ is a closed subspace[32] of H.

6. Let A be a compact linear operator on a Hilbert space. Prove that
$$\mathcal{R}(I - A) = \mathcal{N}(I - A^*)^{\perp}.$$

7. Assume that $x_n \overset{w}{\to} x$ in a Hilbert space H. Moreover assume that $A : H \to H$ is a bounded linear mapping. Does it follow that $Ax_n \overset{w}{\to} Ax$?

8. Show that for every compact operator A on a Hilbert space H there exists a sequence $(A_n)_{n=1}^{\infty}$ in $\mathcal{B}(H, H)$ such that $\dim \mathcal{R}(A_n) < \infty$ for $n = 1, 2, \ldots$ and $A_n \to A$ in $\mathcal{B}(H, H)$.

9. Show that the integral operator on $L^2([0, 1])$ with kernel K satisfying
$$\int_0^1 \int_0^1 |K(x, y)|^2 \, dx dy < \infty$$
is compact[33].

[32] Hint: Let $y \in H$ and suppose $x_n - Ax_n \to y$. Show that one can pick x_n to belong to $\mathcal{N}(I - A)^{\perp}$ for every n. Show that $\{x_n\}$ must be bounded.

[33] Hint: Approximate K by $\tilde{K}(x, y) = \Sigma_{i,j=1}^{n} p_i(x) q_j(y)$. Alternatively approximate K by continuous $\tilde{K}$ and use Arzela-Ascoli's theorem.

10. (a) Suppose $f \in L^1(\mathbb{R})$ and set

$$(Ag)(t) = \int_{-\infty}^{\infty} g(s)f(t-s)\,ds, \quad g \in L^2(\mathbb{R}).$$

Prove that A defines a bounded linear operator on $L^2(\mathbb{R})$ with an operator norm $\leq \|f\|_1$.

(b) Suppose $h > 0$ and set

$$(Bg)(t) = \frac{1}{2h}\int_{t-h}^{t+h} g(s)\,ds, \quad g \in L^2(\mathbb{R}).$$

Prove that B defines a bounded linear operator on $L^2(\mathbb{R})$ with norm 1.

11. Let $k \in L^2([0,\pi] \times [0,\pi])$ and consider the linear mapping

$$T : L^2([0,\pi]) \to L^2([0,\pi])$$

given by

$$Tf(x) = \int_0^{\pi} k(x,y)f(y)\,dy, \quad x \in [0,\pi]$$

for $f \in L^2([0,\pi])$. One standard estimate for the operator norm for T is

$$\|T\| \leq \|k\|_{L^2([0,\pi]\times[0,\pi])}.$$

Prove[34] that also the following estimate is true:

$$\|T\| \leq (\sup_x \int_0^{\pi} |k(x,y)|\,dy)^{\frac{1}{2}} (\sup_y \int_0^{\pi} |k(x,y)|\,dx)^{\frac{1}{2}}.$$

Finally apply these two estimates to the kernel function $k(x,y) = \cos(x-y)$, i.e. calculate the two upper bounds for the operator norm.

12. Set

$$(Ag)(t) = tg(t), \quad g \in L^2([0,1]).$$

Prove that A defines a linear bounded self-adjoint operator on $L^2([0,1])$ without eigenfunctions.

13. Find[35] a mapping $f : [0,1] \to L^2([0,1])$ such that $f(t_1) \neq f(t_2)$ for all $t_1 \neq t_2$ and such that the vectors $f(t_1) - f(t_2)$ and $f(t_3) - f(t_4)$ are orthogonal for all $t_1 < t_2 < t_3 < t_4$.

14. The operator A on $L^2([0,1])$ is defined by

$$(Af)(x) = \int_0^x f(y)\,dy, \quad 0 \leq x \leq 1.$$

Find A^*.

[34] Apply the formula $\|g\| = \sup_{\|h\|=1} |\langle g, h\rangle|$ to Tf. Also the estimate $ab \leq \frac{c}{2}a^2 + \frac{1}{2c}b^2$ for all $a, b \in \mathbb{R}$ and $c > 0$ can come in handy.

[35] Hint: Let $f(t)$ be the characteristic function for the set $[0,t]$ for $t \in [0,1]$.

15. Show that an operator of rank n can have at most n eigenvalues.

16. Set
$$(Ag)(t) = \int_{-\infty}^{\infty} \frac{g(s)}{1+(t-s)^2}\, ds, \quad g \in L^2(\mathbb{R}).$$
Prove that A defines a linear bounded and self-adjoint operator on $L^2(\mathbb{R})$. Finally prove that A is not a compact operator.

17. Set
$$(Tf)(x) = \int_0^{\pi} \sin(x+y) f(y)\, dy, \quad 0 \le x \le \pi.$$
Find the norm of T regarded as an operator on
   (a) the Banach space $C([0,\pi])$
   (b) the Hilbert space $L^2([0,\pi])$.

18. Give an example of a non-self-adjoint operator on a Hilbert space H whose range is H and which is not invertible.

19. Let $T_n : E \to H$, $n = 1, 2, \ldots$, be a sequence of bounded linear operators from a normed space E into a Hilbert space H. We say that
   (a) $(T_n)_{n=1}^{\infty}$ is convergent in $\mathcal{B}(E,H)$ (or convergent in norm in $\mathcal{B}(E,H)$ or uniformly operator convergent) if $(T_n)_{n=1}^{\infty}$ is convergent in $\mathcal{B}(E,H)$;
   (b) $(T_n)_{n=1}^{\infty}$ is strongly operator convergent if $(T_n(x))_{n=1}^{\infty}$ converges in H for all $x \in E$;
   (c) $(T_n)_{n=1}^{\infty}$ is weakly operator convergent if $(T_n(x))_{n=1}^{\infty}$ converges weakly in H for all $x \in E$.

   Show that a) $\Rightarrow$ b) $\Rightarrow$ c). Moreover, let A_n, B_n be operators on l^2 defined by
$$A_n((x_1, x_2, \ldots)) = (\underbrace{0, \ldots, 0}_{n \text{ positions}}, x_{n+1}, x_{n+2}, \ldots)$$
and
$$B_n((x_1, x_2, \ldots)) = (\underbrace{0, \ldots, 0}_{n \text{ positions}}, x_1, x_2, \ldots).$$
In what modes do these sequences of operators converge?

20. A bounded linear operator A on a Hilbert space H is called **unitary** if $A^*A = AA^* = I$. Show that if A is unitary then $\|Ax\| = \|x\|$ for all $x \in H$, i.e. unitary operators do not change lengths. Deduce that all eigenvalues of unitary operators have modulus 1, and eigenvectors belonging to different eigenvalues are orthogonal. Show that all unitary operators are invertible.

   If B is a self-adjoint operator, show that e^{iB} is unitary.

21. A bounded linear operator A on a Hilbert space H is called a **Hilbert-Schmidt operator** if the series $\Sigma_{ij} |\langle Ae_i, f_j\rangle|^2$ converges whenever (e_i) and (f_j) are orthonormal bases for the Hilbert space H. Show that this sum equals $\Sigma_i \|Ae_i\|^2$, and deduce that it is independent of the choice of bases (e_i) and (f_j).

Show that the set of Hilbert-Schmidt operators on a given Hilbert space H is a vector space, and that $\|A\|_{HS} = (\Sigma_i \|Ae_i\|^2)^{1/2}$ is a norm on that space. Show that $\|A\|_{HS} \geq \|A\|$ where $\|A\|$ is the usual operator norm. Give an example in which $\|A\|_{HS} > \|A\|$.

If A and B are Hilbert-Schmidt operators, show that $\Sigma\langle Ae_i, Be_i\rangle$ converges absolutely for every orthonormal basis (e_i), and is independent of the choice of (e_i). Show that one can define an inner product $[A, B]$ on the space of Hilbert-Schmidt operators on H by $[A, B] = \Sigma\langle Ae_i, Be_i\rangle$.

If A and B are integral operators on $L^2([0, 1])$ with continuous kernels K and L respectively, show that they are Hilbert-Schmidt operators, and $[A, B] = \int\int K(s, t)\overline{L(s, t)}\, dsdt$.

22. A bounded linear operator A on a Hilbert space is called **normal** if it commutes with its adjoint, $AA^* = A^*A$. Every self-adjoint operator is obviously normal.

    (a) Show that if the function $K(x, y)$ satisfies $K(x, y) = \overline{K(y, x)}$, then for any real d, the operator $u \mapsto du + i\int_0^1 K(x, y)u(y)\, dy$ on the complex Hilbert space $L^2([0, 1])$ is normal.

    (b) Show that if B, C are commuting self-adjoint operators, then $B + iC$ is normal.

    (c) Prove the converse of (b), i.e. for any normal operator A, there are self-adjoint commuting operators B, C such that $A = B + iC$.

23. Show that a compact normal operator has a complete set of orthogonal eigenvectors.

24. Given an infinite matrix of numbers k_{ij}, $i, j = 1, 2 \ldots$, we say that the double series $\Sigma_{ij}|k_{ij}|^2$ converges if for each i the series $\Sigma_j|k_{ij}|^2$ converges to a number L_i such that $\Sigma_i L_i$ converges, and for each j the series $\Sigma_i|k_{ij}|^2$ converges to a number M_j such that $\Sigma_j M_j$ converges. If $\Sigma_{ij}|k_{ij}|^2$ converges and $k_{ij} = \overline{k_{ji}}$ for all i, j, we define an operator K on the space l^2 by $(Kx)_i = \Sigma_{j=1}^{\infty} k_{ij}x_j$. Show that K is a compact self-adjoint operator $l^2 \to l^2$, and write out what the spectral theorem says in this case.

25. Let (p_i) and (q_i) be two complete orthonormal sets for $L^2([0, 1])$. Let H be the space of square-integrable functions of two variables on the square $0 \leq x, y \leq 1$, with inner product $\int_0^1\int_0^1 f(x, y)\overline{g(x, y)}\, dxdy$.

    (a) Show that the set of functions $p_i(x)q_j(y)$ is orthonormal in H.

    (b) Show that if $\phi \in H$ and $\int_0^1\int_0^1 \phi(x, y)p_i(x)q_j(y)\, dxdy = 0$ for all i, j, then $\phi = 0$.

    (c) The set of functions $p_i(x)q_j(y)$ is labeled by two integers and is therefore countable, and can be arranged in a sequence. Prove that this sequence is a complete orthonormal sequence.

26. Given a function K such that $K(x, y) = \overline{K(y, x)}$ and $\int_0^1\int_0^1 |K(x, y)|^2\, dxdy$ exists, let λ_i and ϕ_i be the eigenvalues and orthonormal eigenfunctions of the integral operator on $L^2([0, 1])$ whose kernel is K. Show that

$$K(x, y) = \Sigma_i \lambda_i \phi_i(x)\overline{\phi_i(y)},$$

the convergence being with respect to the norm in the space H in the previous problem. Show also that

$$\int_0^1\int_0^1 |K(x, y)|^2\, dxdy = \Sigma_i|\lambda_i|^2.$$

27. $K : \mathbb{R}^2 \to \mathbb{C}$ is a piecewise continuous function, and $K(x, y) = \overline{K(y, x)}$. The integral operator A on $L^2([0, 1])$ with kernel K has eigenvalues λ_i and orthonormal eigenfunctions ϕ_i.

   (a) Show that the series $\Sigma c_n \phi_n(x)$ converges absolutely and uniformly if the constants satisfy $\Sigma |c_n/\lambda_n|^2 < \infty$.

   (b) Show that if f is in the range of A, then the series $\Sigma \langle f, \phi_n\rangle \phi_n(x)$ converges absolutely and uniformly to f on $[0, 1]$. Is this still true if we remove the condition that f lies in the range of A?

28. Above it was shown that the eigenvalues λ_i of an integral operator with square-integrable kernel are such that $\Sigma |\lambda_i|^2$ converges. Is this true for compact self-adjoint operators in general?

29. Let T be the linear mapping on $L^2([0, 1])$ defined by

$$Tf(x) = \int_0^1 (x + y)f(y)\, dy, \quad 0 \le x \le 1.$$

   Show that T is bounded and calculate $\|T\|$.

30. Let H be a Hilbert space. Prove or disprove the statement: Every bounded linear mapping on H preserves orthogonality.

31. Let X be a separable Hilbert space and $T : X \to X$ a compact linear operator. Show that T can be approximated by finite rank operators in $\mathcal{B}(H)$, i.e. there exist a sequence of finite rank operators T_n on H such that $T_n \to T$ in operator norm.

32. Let $(e_n)_{n=1}^\infty$ be an ON-basis for a Hilbert space H and assume that $T : H \to H$ is a bounded linear operator on H such that

$$\Sigma_{n=1}^\infty \|Te_n\|^2 < \infty.$$

   Show that if $(f_n)_{n=1}^\infty$ is another ON-basis for H then

$$\Sigma_{n=1}^\infty \|Tf_n\|^2 = \Sigma_{n=1}^\infty \|Te_n\|^2.$$

   Moreover show that

$$\|T\|^2 \le \Sigma_{n=1}^\infty \|Te_n\|^2.$$

33. Set $\mathbb{R}_+ = \{x \in \mathbb{R} : x \ge 0\}$. For $f \in L^2(\mathbb{R}_+)$ define

$$Mf(x) = \frac{1}{x}\int_0^x f(t)\, dt, \quad x > 0.$$

   Show that

$$M : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$$

   is a bounded linear mapping on $L^2(\mathbb{R}_+)$, calculate the operator norm of $I - M$ and, finally, determine the adjoint operator of M. Here I denotes the identity operator on $L^2(\mathbb{R}_+)$.

34. Let X be a Banach space and $T : X \to X$ a compact[36] linear operator. Show that there exists a constant C such that for every $y \in \mathcal{R}(I+T)$ there exists a $x \in X$ with $y = (I+T)x$ such that
$$\|x\| \leq C\|y\|.$$

35. Let a_n, $n = 1, 2, 3, \ldots$ be non-negative reals and set
$$C = \{\not{x} \in l^2 : \not{x} = (x_n)_{n=1}^{\infty},\ |x_n| \leq a_n \text{ all}\, n\}.$$
Show that if C is a compact subset in l^2 then $a_n \to 0$ as $n \to \infty$. For what sequences $(a_n)_{n=1}^{\infty}$ is C compact?

36. Let T be defined on $L^2([0,1])$ by $Tf(x) = \int_0^x f(y)\,dy$. Show that T is a compact operator on $L^2([0,1])$ with $\sigma(T) = \{0\}$. In particular prove that T has no eigenvalues $\neq 0$.

37. Let A be the linear mapping on $L^2([0,1])$ defined by
$$Af(x) = \int_0^1 (x-y)^2 f(y)\,dy, \quad 0 \leq x \leq 1.$$
Calculate
   (a) A^*
   (b) $\|A\|$.

38. Let T be a positive, self-adjoint, compact operator on a Hilbert space H. Show that
$$\langle Tx, x\rangle^n \leq \langle T^n x, x\rangle \cdot \langle x, x\rangle^{2(n-1)},$$
for all positive integers n and all $x \in H$.

39. Let A be the linear mapping on $L^2([0,1])$ defined by
$$Af(x) = \int_0^1 (x-y) f(y)\,dy, \quad 0 \leq x \leq 1.$$
Calculate
   (a) A^*A
   (b) $\|A\|$.

40. Let T be a self-adjoint operator on a Hilbert space H. Assume that T^n is compact for some integer $n \geq 2$. Prove that T is compact.

41. Let H be an infinite-dimensional Hilbert space and let $T : H \to \mathbf{C}$ be a bounded linear functional $\neq 0$. Calculate the dimension for the subspace $\mathcal{N}(T)^{\perp}$ of H. Give an example of a Hilbert space H and a functional T as above.

[36] Exactly the same definition as for a linear operator on a Hilbert space

42. Let T be a self-adjoint, positive, compact operator on a Hilbert space H with $\|T\| \leq 1$. Give an estimate[37] for
$$\|3T^4 - 20T^3 + T^2\|.$$

43. Let S be a dense subset in a Banach space X. Moreover let $(T_n)_{n=1}^{\infty}$ be a sequence of linear operators on X. Assume that

   (a) $\lim_{n\to\infty} T_n x$ exists for every $x \in S$ and

   (b) there exists a $C > 0$ such that
   $$\|T_n x\| \leq C\|x\|$$
   for all n and all $x \in X$.

   Show that $\lim_{n\to\infty} T_n x$ exists for every $x \in X$.

44. For $\mathbf{x} = (\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots) \in l^2$ define
$$(T\mathbf{x})_n = \begin{cases} x_{n+1} + 2x_{n-1} + 10x_n, & n = 2k, k \in \mathbb{Z} \\ 2x_{n+1} + x_{n-1} + 10x_n, & n = 2k+1, k \in \mathbb{Z} \end{cases}.$$

   Which of the statements below hold true?

   (a) T is a bounded linear operator on l^2

   (b) T is self-adjoint

   (c) T is an invertible operator[38].

45. Let T be a bounded linear operator on a Hilbert space H where $\dim \mathcal{R}(T) = 1$. Show that for every $y \in \mathcal{R}(T)$, $y \neq 0$, there exists a uniquely defined $x \in H$ such that
$$Tz = \langle z, x\rangle y, \quad z \in H.$$

   Moreover show that
$$\|T\| = \|x\| \cdot \|y\|.$$

   Apply this fact for calculating the operator norm for the mapping
$$Tf(t) = \int_0^1 e^{t-s} f(s)\, ds, \qquad f \in L^2[0,1].$$

46. Set
$$Au(x) = \int_0^{\pi} e^{x+y} \cos(x+y) u(y)\, dy, \qquad x \in [0,\pi].$$

   Calculate the operator norm for A and see if A is a compact operator on the Banach space

[37] Better than the trivial estimate
$$\|3T^4 - 20T^3 + T^2\| \leq 3\|T\|^4 + 20\|T\|^3 + \|T\|^2 \leq 24.$$

[38] i.e. $T^{-1} \in \mathcal{B}(l^2)$.

(a) $C[0,\pi]$,

(b) $L^2[0,\pi]$.

47. Let T be a normal linear operator on a Hilbert space H, i.e. T is a bounded linear operator that commutes with its adjoint operator T^*, more precisely

$$TT^* = T^*T.$$

Show that

(a) $\|Tx\| = \|T^*x\|$ for all $x \in H$;

(b) λ is an eigenvalue with the eigenvector x for T iff $\overline{\lambda}$ is an eigenvalue with the eigenvector x for T^*.

48. Let $h \in C([0,1] \times [0,1])$ be a real-valued function such that

$$h(x,y) = h(y,x) > 0$$

for all $x, y \in [0,1]$. Set

$$Tf(x) = \int_0^1 h(x,y) f(y)\, dy, \quad x \in [0,1]$$

for $f \in L^2([0,1])$. Show that the bounded linear operator T on $L^2([0,1])$ has an eigenvalue $\lambda = \|T\|$ which is simple.

49. For $u \in C[0,1]$ set

$$(Au)(x) = \int_0^{1-x} |x-y| u(y)\, dy, \quad x \in [0,1].$$

Show that A is a bounded linear operator on the Banach space $C[0,1]$ and calculate the operator norm $\|A\|$.

50. Let H be a complex Hilbert space and A a bounded linear operator on H with the property

$$\langle Ax, x \rangle \in \mathbb{R}$$

for all $x \in H$. Prove that A is self-adjoint.

51. Calculate the operator norm for $A : C[0,\pi] \to C[0,\pi]$ defined by

$$(Af)(x) = \int_0^{\pi} (1 + e^{i(x-y)}) f(y)\, dy.$$

Also calculate the operator norm for $B : L^2[0,\pi] \to L^2[0,\pi]$ defined by

$$(Bf)(x) = \int_0^{\pi} (1 + e^{i(x-y)}) f(y)\, dy.$$

The functions are complex-valued.

52. Let T be defined for $\mathbf{x} = (x_n)_{n=1}^{\infty}$ by

$$(T\mathbf{x})_n = nx_n, \quad n = 1, 2, \ldots$$

Show that $D(T) = \{\mathbf{x} \in l^2 : T\mathbf{x} \in l^2\}$ is a dense subset in l^2 and that T is a bounded operator[39] in l^2, i.e. $\mathbf{x}_n \in l^2$ for $n = 1, 2, \ldots$, $\mathbf{x}_n \to \mathbf{y}$ i l^2, $T\mathbf{x}_n \to \mathbf{z}$ i l^2 implies that $\mathbf{y} \in D(T)$ and $T\mathbf{y} = \mathbf{z}$.

53. Consider the integral operator

$$Af(x) = \int_0^{2\pi} \cos(x-y) f(y)\, dy, \quad 0 \leq x \leq 2\pi.$$

Show that A defines a bounded linear operator on the Banach spaces (real-valued functions)

(a) $C[0, 2\pi]$
(b) $L^2[0, 2\pi]$.

Also calculate the operator norm $\|A\|$ for one of these spaces.

54. Consider the mapping

$$(x_1, x_2, x_3, \ldots) \mapsto (x_1, \frac{1}{2}(x_1 + x_2), \frac{1}{3}(x_1 + x_2 + x_3), \ldots, \frac{1}{n}(x_1 + x_2 + \ldots x_n), \ldots).$$

Show that this is a bounded linear mapping on l^2 that is not surjective.

55. Let T be a bounded linear operator on a Hilbert space H with $\|T\| = 1$. Assume that there exists a $x_0 \in H$ such that $Tx_0 = x_0$. Show that we have $T^* x_0 = x_0$.

[39] Use e.g. the fact that T is a symmetric operator.

6.7 Ordinary differential equations

*Key words:*Green's function, symmetric operators

1. Calculate the Green's functions for the boundary value problems

   a) $\begin{cases} u''(x) + u(x) = f(x) \\ u'(0) = u'\left(\dfrac{\pi}{2}\right) = 0,\ 0 \leq x \leq \dfrac{\pi}{2} \end{cases}$

   b) $\begin{cases} u''(x) = f(x) \\ u(0) - 2u(1) = u'(0) - 2u'(1) = 0,\ 0 \leq x \leq 1 \end{cases}$

   c) $\begin{cases} u''(x) + u(x) = f(x) \\ u(0) = u'(0) = 0,\ 0 \leq x \leq T \end{cases}$

   d) $\begin{cases} \dfrac{1}{6}u^{(4)}(x) = f(x) \\ u(0) = u'(0) = u(1) = u'(1) = 0 \end{cases}$

   e) $\begin{cases} u^{(4)}(x) = f(x) \\ u(0) = u''(0) = u(1) = u''(1) = 0 \end{cases}$

2. Show that (using the notations from "A note on ordinary differential equations") the boundary value problem
$$\begin{cases} Lu = f \\ Ru = c \end{cases}$$
is uniquely solvable for every $f \in C^n(I)$ and $c \in \mathbb{C}^n$ iff
$$\det\{R_j u_k\}_{1 \leq j,k \leq n} \neq 0.$$

3. Show that the Green's function $g(x,t)$ in Example 1 on page 9 in "A note on ordinary differential equations" satisfies $g(x,t) = \overline{g(t,x)}$ and hence the operator $\tilde{G} : L^2([0,1]) \to L^2([0,1])$ defined by
$$(\tilde{G}f)(x) = \int_0^1 g(x,t)f(t)dt,$$
is self-adjoint.

4. Show that the problem
$$\begin{cases} u''(x) + u(x) = e^{ix} + \frac{1}{2}\text{Re } u(x),\ 0 \leq x \leq \pi/2 \\ u'(0) = u'(\pi/2) = 0,\ u \in C^2([0,\pi/2]) \end{cases}$$
has a unique solution.

5. Set $(Lu)(x) = u^{(4)}(x)$, $0 \leq x \leq 1$. Show that L_0 is symmetric if

   (a) $R_1 u = u(0), R_2 u = u'(0), R_3 u = u(1), R_4 u = u'(1)$

   (b) $R_1 u = u(0), R_2 u = u''(0), R_3 u = u(1), R_4 u = u''(1)$.

6. Assume that $(Lu)(x) = -u''(x) + u(x)$, $0 \leq x \leq 1$ and that $R_1 u = u(0) - u(1)$ and $R_2 u = u'(0) - u'(1)$. Show that
   (a) L_0 is bijective
   (b) L_0 has both 1- and 2-dimensional eigenspaces.

7. Assume that $(Lu)(x) = (p(x)u'(x))' - q(x)u(x)$, $a \leq x \leq b$, where $p \in C^1(I)$ and $q \in C(I)$ are real-valued and $p(x) > 0$, $a \leq x \leq b$. Moreover assume that

$$R_1 u = \alpha_{11} u(a) + \alpha_{21} u'(a)$$

and

$$R_2 u = \beta_{12} u(b) + \beta_{22} u'(b)$$

where $(\alpha_{11}, \alpha_{21}) \in \mathbb{R}^2 \setminus \{0\}$ and $(\beta_{12}, \beta_{22}) \in \mathbb{R}^2 \setminus \{0\}$. Show that L_0 is symmetric.

8. Assume that the integral operator

$$(Qf)(x) = \int_a^b q(x,y) f(y) dy, \; a \leq x \leq b,$$

defined on $L^2(I)$ with an L^2- kernel q is self-adjoint and has the eigenvalues $(\lambda_i)_1^\infty$, counted with multiplicity, and corresponding eigenfunctions $(e_i)_1^\infty$.

   (a) Use Bessel's inequality to show that

$$\sum_1^\infty \lambda_i^2 |e_i(x)|^2 \leq \int_a^b |q(x,y)|^2 dy.$$

   (b) Show that

$$\sum_1^\infty \lambda_i^2 \leq \int_a^b \int_a^b |q(x,y)|^2 dx dy.$$

   (c) Show that

$$q(x,y) = \sum_1^\infty \lambda_i e_i(x) \overline{e_i(y)} \quad \text{i } L^2(I \times I).$$

9. Prove that

$$\min(x,y) = \sum_{n=0}^\infty \frac{2}{(n+\frac{1}{2})^2 \pi^2} \sin\left(n + \frac{1}{2}\right)\pi x \sin\left(n + \frac{1}{2}\right)\pi y$$

in $L^2([0,1] \times [0,1])$.

10. Show that the series in Theorem 1.7 in "A note on ordinary differential equations" converges uniformly to u.

11. Prove that there is no function u defined in the interval $[0,1]$ such that

$$\begin{cases} xu'(x) + u(x) = 0, \quad 0 \leq x \leq 1 \\ u(0) = 1. \end{cases}$$

12. Prove the existence of solutions u of the following boundary value problem

$$\begin{cases} -u''(x) = 3(1+u^2(x)), & 0 \le x \le 1 \\ u(0) = u(1) = 0, & u \in C^2([0,1]). \end{cases}$$

13. Prove the existence and uniqueness of solutions of the following boundary value problem

$$\begin{cases} -u''(x) = 7\dfrac{u(x)}{1+u^2(x)} + \sin(\pi x), & 0 \le x \le 1 \\ u(0) = u(1) = 0, & u \in C^2([0,1]). \end{cases}$$

14. Prove the existence and uniqueness of solutions of the following boundary value problem

$$\begin{cases} 4u''(x) = |x + u(x)|, & 0 \le x \le 1 \\ u(0) - 2u(1) = u'(0) - 2u'(1) = 0, & u \in C^2([0,1]). \end{cases}$$

15. Let $\lambda \in \mathbb{R}$ be different from 0.

   (a) Solve the equation

$$\begin{cases} |u'(x)|^2 + \dfrac{1}{\lambda}u''(x) = 1, & 0 \le x \le 1 \\ u(-1) = u(1) = 0, & u \in C^2([0,1]). \end{cases}$$

   (b) Let $u(x) = u(x, \lambda)$ be the solution in part (a). Calculate $\lim_{\lambda \to \pm\infty} u(x, \lambda)$.

16. Show that the following boundary value problem

$$\begin{cases} u''(x) + u(x) = \dfrac{u(x)}{2+u^2(x)}, & 0 \le x \le \dfrac{\pi}{2} \\ u(0) = u(\frac{\pi}{2}) = 0, & u \in C^2([0, \frac{\pi}{2}]) \end{cases}$$

17. Show that the following boundary value problem (almost the same as problem 1)

$$\begin{cases} u''(x) + u(x) = \lambda\dfrac{u(x)}{2+u^2(x)}, & 0 \le x \le \dfrac{\pi}{2} \\ u(0) = u(\frac{\pi}{2}) = 0, & u \in C^2([0, \frac{\pi}{2}]) \end{cases}$$

has a solution for all $\lambda \in \mathbb{R}$.

18. Prove the existence and uniqueness of a solution to the following boundary value problem

$$\begin{cases} u''(x) + u'(x) = \arctan u(x^2), & 0 \le x \le 1 \\ u(0) = u(1) = 0, & u \in C^2([0,1]) \end{cases}$$

19. Consider the differential equation

$$\begin{cases} -u'' = \lambda e^u, & 0 < x < 1, \\ u(0) = u(1) = 0. \end{cases}$$

   (a) Formulate the boundary value problem as a fixed point problem $u = Tu$, where T is an integral operator.

(b) Set $B = \{u \in C([0,1]) : \|u\|_\infty \leq 1\}$. Show that T maps B into itself provided $0 < \lambda < \lambda_0$ for λ_0 sufficiently small. Give a numerical value on λ_0.

(c) Show that the differential equation is uniquely solvable in B with λ chosen as in (b).

20. Show that there exists a unique C^2-function $u(x)$ defined on $[0,1]$ with $u(0) = u(1) = 0$ such that

$$u''(x) - \cos^2 u(x) = 1, \quad x \in [0,1].$$

21. Show that there exists a unique C^2-function $u(x)$ defined on $[0,1]$ such that

$$u(0) - 2u(1) = u'(0) - 2u'(1) = 0$$

and

$$4u''(x) - |u(x) + x| = 0, \quad x \in [0,1].$$

22. Show that there exists a unique C^2-function $u(x)$ defined on $[0,1]$ such that $u(0) = u'(0) = 0$ and

$$u''(x) - u(x) + \frac{1}{2}(1 + u(x^2)) = 0, \quad x \in [0,1].$$

23. Show that there exists a unique C^2-function $u(x)$ defined on $[0, \frac{\pi}{2}]$ such that $u'(0) = u'(\frac{\pi}{2}) = 0$ and

$$u''(x) + u(x) = \frac{1}{2}\sin u(\frac{1}{2}x^2), \quad x \in [0, \frac{\pi}{2}].$$

24. Let H be a Hilbert space. Apply the spectral theorem to find a H-valued solution $u(t)$ to the initial value problem

$$\begin{cases} \dfrac{du}{dt}(t) + Au(t) = 0, \quad t > 0, \\ \\ u(0) = u_0 \in H, \end{cases}$$

where A is a compact self-adjoint positive operator on H. Show that

$$\|u(t)\| \leq \|u_0\|, \quad t \geq 0.$$

25. Let $f \in C([0,1])$ and $\lambda \in \mathbb{R}$. Show that the equation

$$\begin{cases} u''(x) + u'(x) + \lambda|u(x)| = f(x), & x \in [0,1] \\ u(0) = u(1) = 0, & u \in C^2([0,1]) \end{cases}$$

has a unique solution provided $|\lambda|$ is small enough.