TMA401/MMA400 Functional Analysis 2018/2019
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Solutions to some problems in Homework assignment 1

Problem 1 : Let Y be a finite-dimensional subspace in an infinite-dimensional normed space $(E, \|\cdot\|)$. Show that Y is closed.

Solution: (was discussed in detail in class) Consider

$$Y \ni x_n \to x \text{ in } (E, \|\cdot\|).$$

Let $e_1, e_2, \ldots, e_N$, $N = \dim E < \infty$, be a basis for Y. Set

$$\|\alpha_1 e_1 + \alpha_2 e_2 + \ldots + \alpha_N e_N\|_* = \Sigma_{k=1}^N |\alpha_k|.$$

Then $\|\cdot\|$ and $\|\cdot\|_*$ are two norms on Y that are equivalent, i.e. there are $\alpha, \beta > 0$ such that

$$\alpha\|z\|_* \leq \|z\| \leq \beta\|z\|_*, \text{ all } z \in Y$$

(since Y finite-dimensional). With

$$x_n = \alpha_1^{(n)} e_1 + \alpha_2^{(n)} e_2 + \ldots + \alpha_N^{(n)} e_N,\ n = 1, 2, 3, \ldots$$

and since $(x_n)_{n=1}^\infty$ is a Cauchy sequence in $\|\cdot\|_*$ (and also $\|\cdot\|$) and since $(\mathbb{C}, |\cdot|)$ is a Banach space we conclude that there are $\tilde{\alpha}_k \in \mathbb{C}$, $k = 1, 2, \ldots, N$, such that

$$\alpha_k^{(n)} \to \tilde{\alpha}_k,\ \ k = 1, 2, \ldots, N$$

Set

$$\tilde{x} = \tilde{\alpha}_1 e_1 + \tilde{\alpha}_2 e_2 + \ldots + \tilde{\alpha}_N e_N \in Y.$$

Finally we note that $x = \tilde{x}$ since

$$\|x - \tilde{x}\| \leq \|x - x_n\| + \beta \Sigma_{k=1}^N |\alpha_k^{(n)} - \tilde{\alpha}_k| \to 0$$

as $n \to \infty$. Hence $x \in Y$ and Y is closed.

Problem 2 : Consider the normed space $C([0,1])$ with norm $\|f\| = \max_{x \in [0,1]} |f(x)|$. Set $M = \{f \in C([0,1]) :\ f \text{ is an increasing function}\}$. Show that

1. M is not an open set,
2. M is a closed set.

Solution: 1. A function $f : [0,1] \to \mathbb{R}$ is called increasing if for all $x, \tilde{x} \in [0,1]$

$$x < \tilde{x} \quad \Rightarrow \quad f(x) \leq f(\tilde{x}).$$

Hence $\mathbb{0} \in M$, where $\mathbb{0}(x) = 0$ for $x \in [0,1]$. Moreover for all $\epsilon > 0$ we have $g_\epsilon \notin M$ where $g_\epsilon(x) = -\frac{\epsilon}{2}x,\ x \in [0,1]$. Hence $B(\mathbb{0}, \epsilon) \not\subset M$ for $\epsilon > 0$ and so M is not an open set in $C([0,1])$.

2. Consider $M \ni f_n \to f$ in $C([0,1])$ (with the max-norm). Fix arbitrary $x, \tilde{x} \in [0,1]$ with $x < \tilde{x}$. We obtain

$$f(x) - f(\tilde{x}) = \lim_{n\to\infty} f_n(x) - \lim_{n\to\infty} f_n(\tilde{x}) = \lim_{n\to\infty} (f_n(x) - f_n(\tilde{x})) \leq 0.$$

Hence $f \in M$ and so M is a closed set.

Problem 3 : Let $C^1([0,1])$ be the vector space of all continuously differentiable functions $f : [0,1] \to \mathbb{R}$. Show that

1. $C^1([0,1])$ with the norm $|||f||| = \|f\| + \|f'\|$ is a Banach space,
2. $C^1([0,1])$ with the norm $\|f\|$ is not a Banach space.

Here $\|f\|$ denotes $\max_{x\in[0,1]} |f(x)|$.

Solution: 1. Consider a Cauchy sequence $(f_n)_{n=1}^\infty$ in $C^1([0,1])$ with the norm $||| \cdot |||$. Then $(f_n)_{n=1}^\infty$ and $(f'_n)_{n=1}^\infty$ are Cauchy sequences in the Banach space $(C([0,1]), \|\cdot\|)$and hence converges to f, g respectively in $C([0,1])$ with respect to the norm $\|\cdot\|$. It remains to show that $f \in C^1([0,1])$ and $f' = g$. Since

$$f_n(x) - f_n(0) = \int_0^x f'_n(t)\, dt \quad x \in [0,1]$$

and $f'_n \to g$ uniformly on $[0,1]$ ($\|f'_n - g\| \to 0$) we get

$$f(x) - f(0) = \int_0^x g(t)\, dt \quad x \in [0,1]$$

Here the RHS is continuously differentiable and the statement in 1. follows.

2. (for example) Set $f_n(x) = \sqrt{(x-\frac{1}{2})^2 + \frac{1}{n}}$, $n = 1, 2, 3, \ldots$, and $f(x) = |x - \frac{1}{2}|$ for $x \in [0,1]$. Then $f_n \in C^1([0,1])$ for all n, $f \notin C^1([0,1])$ and

$$\|f_n - f\| = \max_{x\in[0,1]} |f_n(x) - f(x)| = \max_{x\in[0,1]} \frac{\frac{1}{n}}{\sqrt{(x-\frac{1}{2})^2 + \frac{1}{n}} + |x - \frac{1}{2}|} \leq \frac{1}{\sqrt{n}}.$$

This shows that $(f_n)_{n=1}^\infty$ is a Cauchy sequence in $C^1([0,1])$ with respect to the norm $\|\cdot\|$ that does not converge.

Problem 4 : Consider the normed space $C([0,1])$ with norm $\|f\| = \max_{x\in[0,1]} |f(x)|$. Assume that $T : C([0,1]) \to C([0,1])$ is a linear mapping with the property that $Tf(x) \geq 0$ for all $x \in [0,1]$ provided $f(x) \geq 0$ for all $x \in [0,1]$. Show that

1. T is continuous,
2. $\|T\| = \max_{x\in[0,1]} T\mathbb{1}(x)$ where $\mathbb{1}$ denotes the constant function taking the value 1.

Solution: Since T is a linear mapping $C([0,1]) \to C([0,1])$ it is equivalent to show that T is bounded. Fix an arbitrary $f \in C([0,1])$. Then

$$\|f\|\mathbb{1}(x) \pm f(x) \geq 0 \text{ for all } x \in [0,1]$$

and hence

$$T(\|f\|\mathbb{1} \pm f)(x) \geq 0 \text{ for all } x \in [0,1]$$

which yields

$$-T(\mathbb{1})(x)\|f\| \leq T(f)(x) \leq T(\mathbb{1})(x)\|f\| \text{ for all } x \in [0,1].$$

Hence

$$\|T(f)\| = \max_{x\in[0,1]} |T(f)(x)| \leq \max_{x\in[0,1]} T(\mathbb{1})(x)\|f\|$$

where we note that $0 \leq T(\mathbb{1})(x) \leq \max_{x\in[0,1]} T(\mathbb{1})(x) < \infty$. Since $\mathbb{1} \in C([0,1])$ we have $\|T\| = \max_{x\in[0,1]} T(\mathbb{1})(x)$.