TMA401/MMA400 Functional Analysis 2018/2019

Peter Kumlin
Mathematics
Chalmers & GU

Homework assignment 2

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Problem 1: Suppose that K is a compact subset in a Banach space $(E, \|\cdot\|)$ and that $T : K \to K$ with the property

$$\|T(x) - T(\tilde{x})\| < \|x - \tilde{x}\| \text{ for all } x, \tilde{x} \in K,\ x \neq \tilde{x}.$$

It was proven in class that T has a unique fixed point. Show that for every $x_0 \in K$ the sequence $(T^n(x_0))_{n=1}^{\infty}$ converges to the fixed point for T.

Problem 2: Assume that $\mathcal{M}$ is a subset of $C([0,1])$ such that there are constants $m, L > 0$ such that $|f(\frac{1}{2})| \leq m$ for all $f \in \mathcal{M}$ and

$$|f(x) - f(\tilde{x})| \leq L|x - \tilde{x}| \text{ for all } f \in \mathcal{M} \text{ and all } x, \tilde{x} \in [0,1].$$

Show that $\mathcal{M}$ is relatively compact in $C([0,1])$.

Problem 3: Let $(E, \|\cdot\|)$ be a normed space. Set

$$T(x) = \begin{cases} x & \text{if } \|x\| \leq 1, \\ \frac{1}{\|x\|}\, x & \text{if } \|x\| > 1. \end{cases}$$

1. Show that $\|T(x) - T(\tilde{x})\| \leq 2\|x - \tilde{x}\|$ for all $x, \tilde{x} \in E$.
2. Show that the constant 2 cannot be improved in general. Take $E = \mathbb{R}^2$ with $\|\cdot\| = \|\cdot\|_{l^1}$.
3. What can be said if E is a Hilbert space with the norm given by the inner product?

Problem 4: Let $T : \{\mathbb{x} \in \mathbb{R}^n : \|\mathbb{x}\| \leq 1\} \to \mathbb{R}^n$ be a continuous mapping. Moreover assume that $\langle T(\mathbb{x}), \mathbb{x}\rangle > 0$ for all $\mathbb{x}$ with $\|\mathbb{x}\| = 1$. Here $\langle\cdot,\cdot\rangle$ denotes the standard inner product on $\mathbb{R}^n$ with the induced norm $\|\cdot\|$. Prove[1] that there exists a $\mathbb{x}_0 \in \{\mathbb{x} \in \mathbb{R}^n : \|\mathbb{x}\| \leq 1\}$ such that $T(\mathbb{x}_0) = \mathbb{0}$.

[1] Hint: Consider the mapping $G(\mathbb{x}) = \mathbb{x} - \epsilon T(\mathbb{x})$ for some properly choosen $\epsilon > 0$.