TMA401/MMA400 Functional Analysis 2018/2019
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Homework assignment 3

Deadline 2018-10-18

Problem 1: Set
$$A(f)(x) = \int_0^1 (x-y)\, f(y)\, dy, \quad 0 \le x \le 1.$$
Find the operator norm for A regarded as an operator on

1. the Banach space $C([0,1])$,
2. the Hilbert space $L^2([0,1])$.

Both spaces are equipped with the standard norms.

Problem 2: Let E be a complex Hilbert space. Suppose that $A \in \mathcal{B}(E,E)$ with
$$\langle A(x), x \rangle \ge 0 \text{ for all } x \in E.$$
Show that A is self-adjoint.

Problem 3: Let E be a complex Hilbert space with a complete ON-sequence $(e_n)_{n=1}^{\infty}$. Let $(\alpha_n)_{n=1}^{\infty}$ be a sequence of complex numbers and set
$$A(x) = \Sigma_{n=1}^{\infty} \alpha_n \langle x, e_n \rangle e_n, \; x \in E.$$
Give necessary and sufficient conditions on the sequence $(\alpha_n)_{n=1}^{\infty}$ such that

1. $A(x) \in E$ for every $x \in E$,
2. $A \in \mathcal{B}(E,E)$ and give an expression for $\|A\|$ in terms of the α_n:s,
3. A self-adjoint,
4. $A \in \mathcal{K}(E,E)$ (i.e. A compact),
5. A 1-1.

All statements in the five cases must be proved.

Problem 4: Analyse existence and uniqueness for solutions to the following BVP:
$$\begin{cases} u''(x) - 2u'(x) + u(x) + \alpha \dfrac{1}{1+(u(x^2))^2} = 0, & 0 \le x \le 1 \\ u(0) = 2, u(1) = 1, & u \in C^2([0,1]) \end{cases}$$
Here α is a real number and all functions are real-valued. What can be said for different values of α? Note that the boundary conditions are not homogeneous.