Written exam TMA401/MMA400, 2013-10-23 (sketches)

1. $(*)\begin{cases} u''(x) + u(x) = -\lambda \cos(1+u(x)) & x \in [0,1] \\ u(0) = u(1) = 0 \end{cases}$

Show that $(*)$ has a unique C^2-solution u for $|\lambda| < \varepsilon$, $\varepsilon > 0$ small and that there exists a C^2-solution for arbitrary $\lambda \in \mathbb{R}$.

Solution. (I) calculation of the Greens function.

$Lu = u'' + u$. $N(L)$ has a basis $u_1(x) = \sin x$, $u_2(x) = \cos x$

Set $g(x,t) = (a_1(t)u_1(x) + a_2(t)u_2(x))\theta(x-t) + b_1(t)u_1(x) + b_2(t)u_2(x)$

where

$$\begin{cases} a_1(t)\sin t + a_2(t)\cos t = 0 \\ a_1(t)\cos t - a_2(t)\sin t = 1 \end{cases}$$

and from the BC:s

$$\begin{cases} b_2(t) = 0 \\ a_1(t)\sin 1 + a_2(t)\cos 1 + b_1(t)\sin 1 + b_2(t)\cos 1 = 0 \end{cases} \quad 0 < t < 1$$

This yields

$$\begin{cases} a_1(t) = \cos t \\ a_2(t) = -\sin t \end{cases} \qquad \begin{cases} b_2 = \dfrac{\sin(t-1)}{\sin 1} \\ b_2(t) = 0 \end{cases}$$

and hence we obtain

$$g(x,t) = \sin(x-t)\,\theta(x-t) + \frac{\sin(t-1)}{\sin 1}\sin x$$

(II) Set $T(u)(x) = \int_0^1 g(x,t)(-\lambda)\cos(1+u(t))\,dt \quad x \in [0,1]$

for $u \in C([0,1])$. $(C([0,1]), \|\cdot\|)$ is a Banach space if $\|\cdot\|$ denotes the max-norm. By Banach's fixed point theorem we conclude that T has a unique fixed point if T is a contraction, and hence $(*)$ has a unique C^2-solution u. For $u, v \in C([0,1])$

$$|(T(u) - T(v))(x)| = |\lambda|\left|\int_0^1 g(x,t)\,(\cos(1+u(t)) - \cos(1+v(t)))\,dt\right| \le$$

$$\le |\lambda| \int_0^1 |g(x,t)|\,|\cos(1+u(t)) - \cos(1+v(t))|\,dt \le$$

$\leq \{\text{mean value term}\} \leq |\lambda| \int_0^1 |g(x,t)|\,dt \, \|u - v\|$

Hence T is a contraction if $|\lambda| \max_{x\in[0,1]} \int_0^1 |g(x,t)|\,dt < 1$.

A very crude estimate for $|\lambda|$ is $|\lambda| < \frac{\sin 1}{1+\sin 1}$.

(III) For arbitrary $\lambda \in \mathbb{R}$ we have to use Schauder's fixed point theorem. One easily shows that $T : C([0,1]) \to C([0,1])$ is a continuous mapping and that for every sequence $(u_n)_{n\geq 1}^{\infty}$ in $C([0,1])$, $\{T(u_n) : n = 1, 2, \ldots\}$ is (uniformly) bounded and equicontinuous. The statement in the problem follows

(2) Consider $(\ell^1, \|\cdot\|_*)$ where $\|x\|_* = 2\left|\sum_{n=1}^{\infty} x_n\right| + \sum_{n=2}^{\infty}\left(1+\frac{1}{n}\right)|x_n|$

Show that $(\ell^1, \|\cdot\|_*)$ is a Banach space. Are $\|\cdot\|_{\ell^1}$, $\|\cdot\|_*$ equivalent norms?

Solution: It is easy to show that $\|\cdot\|_*$ defines a norm on ℓ^1. To show that $(\ell^1, \|\cdot\|_*)$ is a Banach space it is enough to show that the norms $\|\cdot\|_{\ell^1}$ and $\|\cdot\|_*$ are equivalent. We note that for $x = (x_1, x_2, \ldots, x_n, \ldots) \in \ell^1$

$$\|x\|_* \leq 2\|x\|_{\ell^1} + \frac{3}{2}\|x\|_{\ell^1} = \frac{7}{2}\|x\|_{\ell^1}$$

$$\|x\|_{\ell^1} = |x_1| + \sum_{n=2}^{\infty} |x_n| =$$

$$= \left|\sum_{n=1}^{\infty} x_n - \sum_{n=2}^{\infty} x_n\right| + \sum_{n=2}^{\infty} |x_n| \leq$$

$$\leq \left|\sum_{n=1}^{\infty} x_n\right| + \sum_{n=2}^{\infty} |x_n| + \sum_{n=2}^{\infty} |x_n| \leq 2\|x\|_*$$

This shows that the norms are equivalent and since $(\ell^1, \|\cdot\|_{\ell^1})$ is a Banach space, also $(\ell^1, \|\cdot\|_*)$ is a Banach space

(3) $(e_n)_{n=1}^{\infty}$ ON-basis in Hilbert space H and set $f_n = e_{n+1} - e_n$, $n = 1, 2, \ldots$. Show that $\operatorname{span}\{f_n : n = 1, 2, \ldots\}$ is dense in H

Solution. Set $S = \overline{\operatorname{span}\{f_n : n = 1, 2, \ldots\}}$. Show that $S^{\perp} = \{0\}$

If $S^\perp \setminus \{0\} \neq \phi$ pick $x \in S^\perp \setminus \{0\}$. Since $x \in S^\perp$ we obtain $\langle x, f_n \rangle = 0$ i.e. $\langle x, e_{n+1} \rangle = \langle x, e_n \rangle$ all n. But $(e_n)_{n=1}^\infty$ is an ON-basis and hence by Parseval $\sum_{n=1}^\infty |\langle x, e_n \rangle|^2 = \|x\|^2 < \infty$. Hence $\langle x, e_n \rangle = 0$ all n and Parseval yields $x = 0$. Contradiction. So $S = H$.

④, ⑤ See textbook

⑥ A self-adjoint operator on Hilbert space H and $\lambda \in \mathbb{C}$.
Show $\lambda \in \rho(A) \iff \exists c > 0 : \|x\| \leq c \|A(x) - \lambda x\|$ all $x \in H$

Proof:

$\Rightarrow$: trivial since $\lambda \in \rho(A) \Rightarrow (\lambda I - A)^{-1} \in \mathcal{B}(H, H)$ and

$$x = (\lambda I - A)^{-1} (\lambda I - A)(x), \quad \text{which gives}$$

$$\|x\| \leq \underbrace{\|(\lambda I - A)^{-1}\|_{H \to H}}_{< \infty} \|A(x) - \lambda x\| \quad \text{all } x \in H$$

$\Leftarrow$: Remains to prove $\mathcal{R}(A - \lambda I) = H$ since $(A - \lambda I)(x_1) = (A - \lambda I)(x_2) \Rightarrow x_1 = x_2$ from the assumption $\exists c > 0 \; \|x\| \leq c \|A(x) - \lambda x\|$ all $x \in H$, which also then gives $(\lambda I - A)^{-1} \in \mathcal{B}(H, H)$.

Note that $\mathcal{R}(A - \lambda I)$ is a subspace in H.

$\mathcal{R}(A - \lambda I)$ is closed since if $\mathcal{R}(A - \lambda I) \ni y_n \longrightarrow y$ in H then $y \in \mathcal{R}(A - \lambda I)$ since, pick $x_n \in H$ s.th. $(A - \lambda I)(x_n) = y_n$, $n = 1, 2, \ldots$. We have

$$\|x_n - x_m\| \leq c \|y_n - y_m\| \longrightarrow 0, \quad n, m \to \infty$$

and hence $(x_n)_{n=1}^\infty$ converges in H since H Hilbert. Call the limit x. Hence

$$(A - \lambda I)(x) \longleftarrow (A - \lambda I)(x_n) = y_n \longrightarrow y \quad \text{in } H$$

and $(A - \lambda I)(x) = y$ so $y \in \mathcal{R}(A - \lambda I)$.

Assume now $\mathcal{R}(A - \lambda I)^\perp \setminus \{0\}$ and get a contradiction. Fix $z \in \mathcal{R}(A-\lambda I)^{\perp} \setminus \{0\}$.

$\langle (A-\lambda I)(x), z \rangle = 0$ all $x \in H$ $\iff$

$\langle x, (A^* - \bar{\lambda} I) z \rangle = 0$ all $x \in H$ $\iff$

$\langle x, (A - \bar{\lambda} I)(z) \rangle = 0$ all $x \in H$ $\iff$

$A(z) = \bar{\lambda} z$.

A self-adjoint so all eigenvalues must be real.

If $\lambda \in \mathbb{R}$ we have

$\|z\| \leq \| (A-\lambda I)(z) \| = 0$

which implies $z = 0$ and yields a contradiction.

Done