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Written exam: Functional Analysis TMA401/MMA400
Solutions
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1. Show that the following boundary value problem

$$\begin{cases} u''(x) + u(x) = \dfrac{u(x)}{2+u^2(x)}, \quad x \in [0, \dfrac{\pi}{2}], \\ \\ u(0) = u(\frac{\pi}{2}) = 0, \; u \in C^2([0, \frac{\pi}{2}]). \end{cases}$$

has a unique solution u.

(4p)

Solution: The solution is given in two steps.

1. Determine the Green's function for $\begin{cases} u'' + u = F \\ u(0) = u(\frac{\pi}{2}) = 0 \end{cases}$:
   Set $e(x,t) = a_1(t)\cos x + a_2(t)\sin x$, where $e(t,t) = 0$ and $e'_x(t,t) = 1$. This gives $e(x,t) = \sin(x-t)$. The Green's function takes the form
   $$g(x,t) = \sin(x-t)\theta(x-t) + b_1(t)\cos x + b_2(t)\sin x.$$
   Here $g(0,t) = g(\frac{\pi}{2}, t) = 0$ for $0 < t < \frac{\pi}{2}$ implies that
   $$g(x,t) = \sin(x-t)\theta(x-t) - \sin x \cos t =$$
   $$= \begin{cases} -\cos x \sin y, & 0 \le t \le x \le \frac{\pi}{2} \\ -\sin x \cos t, & 0 \le x \le t \le \frac{\pi}{2} \end{cases}$$
   We see that $g(x,t) \le 0$ for all $x, t \in [0, \frac{\pi}{2}]$.
2. Set
   $$\begin{cases} (Tu)(x) = \int_0^{\frac{\pi}{2}} g(x,t)\frac{u(t)}{2+u^2(t)}dt, & 0 \le x \le \frac{\pi}{2} \\ u \in C([0, \frac{\pi}{2}]) \end{cases}$$
   The boundary value problem has a unique solution iff $T : C([0, \frac{\pi}{2}]) \to C([0, \frac{\pi}{2}])$ has a unique fixed point in the Banach space $(C([0, \frac{\pi}{2}], \|\cdot\|)$, where $\|f\| = \max_{x \in [0, \frac{\pi}{2}]} |f(x)|$. For $u, v \in C([0, \frac{\pi}{2}])$ we get
   $$|(Tu)(x) - (Tv)(x)| \le \int_0^{\frac{\pi}{2}} |g(,t)|| \frac{u(t)}{2+u^2(t)} - \frac{v(t)}{2+v^2(t)}|dt \le$$
   $$\le \{\text{mean value theorem}\} \le$$
   $$\le \frac{1}{2}\int_0^{\frac{\pi}{2}} |g(x,t)|dt\|u-v\| \le \frac{\pi}{4}\|u-v\|$$

This shows that T is a contraction on the space $C([0, \frac{\pi}{2}])$ and the conclusion follows from Banach's fixed point theorem.

2. Let A be a positive compact self-adjoint operator on a Hilbert space H with operator norm ≤ 1. Give an upper estimate for the operator norm of $3A^4 - 20A^3 + A^2$ (better than the trivial estimate 24).

(3p)

Solution: From the Hilbert-Schmidt theorem it follows that there exist an ON-sequence $(e_n)_{n=1}^N$ of eigenvectors corresponding to the non-zero eigenvalues $(\lambda_n)_{n=1}^N$ such that

$$\begin{cases} x = \Sigma_{n=1}^N \langle x, e_n \rangle e_n + u, \ u \in \mathcal{N}(A) \\ A(x) = \Sigma_{n=1}^N \lambda_n \langle x, e_n \rangle e_n \end{cases}$$

since A is a compact self-adjoint operator. Here N is either an integer or ∞. We obtain

$$(3A^4 - 20A^3 + A^2)(x) = \Sigma_{n=1}^N (3\lambda_n^4 - 20\lambda_n^3 + \lambda_n^2)\langle x, e_n \rangle e_n$$

and

$$\|3A^4 - 20A^3 + A^2\| = \sup_{\|x\|=1} \|(3A^4 - 20A^3 + A^2)(x)\| =$$

$$= \sup_{\Sigma_{n=1}^N |\langle x, e_n \rangle|^2 = 1} (\Sigma_{n=1}^N |3\lambda_n^4 - 20\lambda_n^3 + \lambda_n^2|^2 |\langle x, e_n \rangle|^2)^{\frac{1}{2}} \leq$$

$$\leq \sup_{0 \leq \lambda \leq 1} |3\lambda^4 - 20\lambda^3 + \lambda^2|.$$

Here we have used the fact that A is positive, which implies $\lambda_n \geq 0$ for alla n, and $\|A\| \leq 1$, which implies $|\lambda_n| \leq 1$ for alla n, together with Parseval´s formula. It remains to prove that $\sup_{0 \leq \lambda \leq 1} |3\lambda^4 - 20\lambda^3 + \lambda^2| < 24$ (a standard freshman calculus), which is left as an exercise.

3. Let k be a non-zero continuous function on $[-\pi, \pi]$ and define the operator $T \in \mathcal{B}(L^2([-\pi, \pi]))$ by $Tf(x) = k(x)f(x)$. Show that T is not compact.

(4p)

Solution: To show that T is not a compact operator on $L^2([-\pi, \pi])$ it is enuogh to find a sequence $(u_n)_{n=1}^\infty$ in $L^2([-\pi, \pi])$ such that

$$u_n \rightharpoonup \mathbf{0} \ \text{ in } L^2([-\pi, \pi])$$

and

$$T(u_n) \not\to \mathbf{0} \ \text{ in } L^2([-\pi, \pi]).$$

Take u_n to be the function e^{inx}, $x \in [-\pi, \pi]$, n=1,2,3,... Clearly these functions form an ON-sequence in $L^2([-\pi, \pi])$ and hence converges weakly to $\mathbf{0}$ in $L^2([-\pi, \pi])$ but

$$\|Tu_n\| \geq \min_{x \in [-\pi,\pi]} |k(x)| > 0 \text{ for all } n.$$

Hence

$$T(u_n) \not\to \mathbf{0} \text{ in } L^2([-\pi, \pi]).$$

4. State and prove Banach´s fixed point theorem.

(5p)

Solution: See textbook

5. Let $k(x, y) \in C([0, 1] \times [0, 1])$ and define

$$Af(x) = \int_0^1 k(x, y) f(y)\, dy, \quad x \in [0, 1].$$

Show that A is a compact operator on $L^2([0, 1])$ and also on $C([0, 1])$.

(5p)

Solution: (sketch) We note that

$$A(f) \in L^2([0, 1]) \text{ for each } f \in L^2([0, 1])$$

and also

$$A(f) \in C([0, 1]) \text{ for each } f \in C([0, 1]).$$

Also A is linear as an operator on both $L^2([0, 1])$ and $C([0, 1])$. Furthermore

$$\|A\|_{L^2([0,1]) \to L^2([0,1])} \leq \|k\|_{L^2([0,1]\times[0,1])},$$

where we used Hölder´s inequality, and

$$\|A\|_{C([0,1]) \to C([0,1])} \leq \max_{x \in [0,1]} \int_0^1 |k(x, y)|\, dy \leq \max_{(x,y) \in [0,1]\times[0,1]} |k(x, y|.$$

Here $C([0, 1])$ is equipped with the max-norm denoted by $\|f\| = \max_{x \in [0,1]} |f(x)|$.

A is compact operator on L^2:

Set $A_n(f)(x) = \int_0^1 k_n(x, y)\, dy$ for $f \in L^2$ and $x \in [0, 1]$, where

$$k_n(x, y) = \Sigma_{i=1}^n \Sigma_{j=1}^n k(\frac{i}{n}, \frac{j}{n}) \chi_{I_i}(x) \chi_{I_j}(y)$$

and

$$\chi_{I_i}(t) = \begin{cases} 1 & t \in I_i \equiv [\frac{i-1}{n}, \frac{i}{n}) \\ 0 & \text{otherwise} \end{cases} \quad i = 1, 2, 3, \ldots$$

Then $\dim \mathcal{R} A_n \leq n$, i.e. A_n is a sequence of finte-rank operators on the Hilbert space L^2 and hence all A_n are compact. Moreover

$$\|A - A_n\| \leq \|k - k_n\|_{L^2([0,1]\times[0,1])} \to 0 \text{ as } n \to \infty$$

since k is a continuous function on the compact set $[0,1]\times[0,1]$ and hence uniformly continuous. Finally since the vector space of compact linear operators on L^2 is a closed set in the vector space of bounded linear operators on L^2 we conclude that A is compact.

A is compact operator on C:

Here we will fix an arbitrary bounded sequence $(f_n)_{n=1}^{\infty}$ in $C([0,1])$ and prove that there exists a converging subsequence of $(A(f_n))_{n=1}^{\infty}$ in $C([0,1])$. To show this we apply the Arzela-Ascoli theorem. Let $M > 0$ be constant such that

$$\|f_n\| \leq M \quad n = 1, 2, 3, \ldots$$

Then

$$\|A(f_n)\| \leq M \max_{(x,y)\in[0,1\times[0,1]} |k(x,y|, \quad n = 1, 2, 3, \ldots$$

Hence $(A(f_n))_{n=1}^{\infty}$ is uniformly bounded in $C([0,1])$. Remains to prove that $(A(f_n))_{n=1}^{\infty}$ is equicontinuous. This follows from the fact that k is a contionuous function on the compact set $[0,1] \times [0,1]$ and hence uniformly continuous. By AA theorem $(A(f_n))_{n=1}^{\infty}$ has a subsequence that converges in $C([0,1])$.

6. Let $C_n, n = 1, 2, 3, \ldots$, be a sequence of closed convex subsets in a Hilbert space H. Moreover assume that

$$C_1 \supset C_2 \supset \ldots \supset C_n \supset \ldots$$

and that

$$C = \bigcap_{n=1}^{\infty} C_n \neq \emptyset.$$

For $x \notin C$, let $x_n \in C_n$ be defined by

$$\|x - x_n\| = \inf_{y \in C_n} \|x - y\|$$

for $n = 1, 2, 3, \ldots$. Show that $x_n \to \tilde{x}$ in H and give a geometric interpretation of $\tilde{x}$.

(4p)

Solution: Set $d(x, C_n) = \inf_{y\in C_n} \|x - y\|$ for $n = 1, 2, 3, \ldots$ and $d(x, C) = \inf_{y\in C} \|x - y\|$. We know from the assumptions in the problem and the proposition on ¨the closest point property¨ that

(a) $\|x - x_n\| = d(x, C_n)$ for $n = 1, 2, 3, \ldots$

(b) $0 \leq d(x, C_m) \leq d(x, C_n) \leq d(x, C) < \infty$ for all $n > m$ since C is closed convex

(c) $d(x, C) > 0$ since $x \notin C$

By the parallelogram law for the norm in a Hilbert space it follows that

$$\|x_n - x_m\|^2 + \|(x - x_n) + (x - x_m)\|^2 = 2((d(x, C_n))^2 + (d(x, C_m))^2)$$

and hence for $n > m$

$$\|x_n - x_m\|^2 = 2((d(x, C_n))^2 + (d(x, C_m))^2 - 2\|x - \frac{1}{2}(x_n + x_m)\|^2) \leq$$

$$\leq 2((d(x, C_n))^2 + (d(x, C_m))^2 - 2(d(x, C_m))^2).$$

As $\lim_{n\to\infty} d(x, C_n)$ exists we obtain $\|x_n - x_m\| \to 0$ as $n, m \to \infty$. This implies that $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence in the Hilbert space H and hence converges. Call the limit point $\tilde{x}$. We note that $\tilde{x} \in C$ since C_m is closed and $x_n \in C_m$ for all $n \geq m$ and hence $\tilde{x} \in C_m$.

It remains to give a geometric interpretation of $\tilde{x}$. We observe that

$$d(x, C) \leq \|x - \tilde{x}\| = \lim_{n\to\infty} \|x - x_n\| = \lim_{n\to\infty} d(x, C_n) \leq d(x, C).$$

This gives

$$\|x - \tilde{x}\| = d(x, C)$$

and since the closest point in C to x is uniquely defined we have that $\tilde{x}$ must be this point.