Institutionen för matematiska vetenskaper
Chalmers tekniska högskola och Göteborgs universitet

Written exam: Functional Analysis TMA401/MMA400
Solutions (first sketches)
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1. Consider the differential operator $L = (\frac{d}{dx})^2$ defined on $C^2([0,1])$ with boundary conditions $u(0) = u'(1)$ and $u'(0) = u(1)$. Calculate

   (a) the corresponding Green´s function $g(x,t)$, and

   (b) prove that the boundary value problem

$$\begin{cases} u''(x) = \sin(\sqrt{|u(x^2)| + 1}), & x \in [0,1], \\ u(0) = u'(1), u'(0) = u(1) \end{cases}$$

   has a unique solution $u \in C^2([0,1])$.

(4p)

Solution:

Green´s function $g(x,t)$**:** We observe that $u_1(x) = 1$ and $u_2(x) = x$ form a basis for $\mathcal{N}(L)$ where $Lu = u''$. Set

$$g(x,t) = \theta(x-t)(a_1(t)u_1(x) + a_2(t)u_2(x)) + b_1(t)u_1(x) + b_2(t)u_2(x)$$

where

$$\begin{cases} a_1(t) + a_2(t) = 0 \\ a_2(t) = 1 \end{cases}$$

and

$$\begin{cases} b_1(t) = a_2(t) + b_2(t) \\ b_2(t) = a_1(t) + a_2(t) + b_1(t) + b_2(t) \end{cases}$$

We obtain

$$a_1(t) = -t,\ a_2(t) = 1$$

and

$$b_1(t) = t - 1,\ b_2(t) = t - 2.$$

This gives

$$g(x,t) = \theta(x-t)(x-t) + t - 1 + (t-2)x,\ 0 \le x,t \le 1.$$

Unique solution for the BVP: The problem can be rewritten as

$$u(x) = \int_0^1 g(x,t)(\sin(\sqrt{|u(t^2)|+1}))\,dt,\ 0 \le x \le 1.$$

Set

$$T(u)(x) = \int_0^1 g(x,t)(\sin(\sqrt{|u(t^2)|+1}))\,dt,\ 0 \le x \le 1,$$

where $u \in C([0,1])$. Clearly $T : C([0,1]) \to C([0,1])$. We assume that $C([0,1])$ is equipped with the max-norm, i.e. $\|f\| = \max_{x\in[0,1]} |f(x)|$, which makes $C([0,1])$ into a Banach space. From Banach´s fixed point theorem it follows that the BVP above has a unique solution u if T is a contraction on $C([0,1])$. For $u, v \in C([0,1])$ we get

$$|T(u)(x) - T(v)(x)| = |\int_0^1 g(x,t)(\sin(\sqrt{|u(t^2)|+1}) - \sin(\sqrt{|v(t^2)|+1}))\,dt| \le$$

$$\le \int_0^1 |g(x,t)| \cdot |\sin(\sqrt{|u(t^2)|+1}) - \sin(\sqrt{|v(t^2)|+1})|\,dt.$$

Apply the mean value theorem to obtain

$$|\sin(\sqrt{|u(t^2)|+1}) - \sin(\sqrt{|v(t^2)|+1})| \le |\sqrt{|u(t^2)|+1} - \sqrt{|v(t^2)|+1}| \le$$

$$\le \frac{1}{2}|u(t^2) - v(t^2)| \le \frac{1}{2}\|u-v\|.$$

This yields

$$\|T(u) - T(v)\| \le \frac{1}{2}\max_{x\in[0,1]}\int_0^1 |g(x,t)|\,dt \cdot \|u-v\|.$$

By inspection we see that $g(x,t) \le 0$ for all $0 \le x, t \le 1$ so

$$j(x) \equiv \int_0^1 |g(x,t)|\,dt = \int_0^1 g(x,t)(-1)\,dt,\ 0 \le x \le 1$$

will satisfy $j''(x) = -1,\ j(0) = j'(1),\ j'(0) = j(1)$. A calculation gives

$$j(x) = \frac{1}{2} + \frac{3}{2}x - \frac{1}{2}x^2$$

and

$$\frac{1}{2} \le j(x) \le \frac{3}{2},\ 0 \le x \le 1.$$

Finally we have proved

$$\|T(u) - T(v)\| \le \frac{3}{4}\|u-v\|,\ u, v \in C([0,1])$$

and the result follows.

2. For $f \in L^2([0,1])$ and $x \in [0,1]$ set

$$A(f)(x) = \int_0^1 (x-y)f(y)\,dy.$$

Show that

(a) $A(f) \in L^2([0,1])$ for $f \in L^2([0,1])$

(b) A is a bounded linear operator on $L^2([0,1])$, and

(c) calculate $\|A\|$ and also

(d) $\|(A - \frac{1}{2\sqrt{3}}I)^{10}\|$ where I denotes the identity operator on L^2.

(5p)

Solution: Fix $f \in L^2([0,1])$. Using Hölder´s inequality (or Schwartz´ inequality) we get

$$\|A(f)\|_{L^2}^2 = \int_0^1 |\int_0^1 (x-y)f(y)\,dy|^2\,dx \le \int_0^1 (\int_0^1 |x-y||f(y)|\,dy)^2\,dx \le$$

$$\le \int_0^1 (\int_0^1 (x-y)^2\,dy)(\int_0^1 |f(y)|^2\,dy)\,dx = \int_0^1 \int_0^1 (x-y)^2\,dydx \cdot \|f\|_{L^2}^2.$$

This proves (a). (b) is a (boring and) straight-forward calculation. We observe that A is a compact operator and that iA is a self-adjoint (and compact) operator. This implies that

$$\|A\| = \|iA\| = \max_{\lambda \text{ eigenvalue to } iA} |\lambda|.$$

We calculate the eigenvalues to A. Here $A(f)(x) = \lambda f(x)$ implies that $f(x) = a + bx$ and so

$$\lambda(a+bx) = \int_0^1 (x-y)(a+by)\,dy = -\frac{a}{2} - \frac{b}{3} + (a + \frac{b}{2})x.$$

This implies

$$\lambda = \pm i \frac{1}{2\sqrt{3}}$$

and we obtain $\|A\| = \frac{1}{2\sqrt{3}}$. For (d) we note that iA is a compact self-adjoint operator on L^2, which is an infinite-dimensional separable Hilbert space and hence by Hilbert-Schmidt theorem there exists a complete ON-sequence $(e_n)_{n=1}^{\infty}$ of eigenvectors (eigenfunctions) to iA corresponding to the eigenvalues $(\lambda_n)_{n=1}^{\infty}$. WLOG we have $\lambda_1 = \frac{1}{2\sqrt{3}}$, $\lambda_2 = -\frac{1}{2\sqrt{3}}$ and $\lambda_n = 0$ for $n = 3, 4, \ldots$ This gives

$$A(f) = -i\Sigma_{n=1}^{\infty} \lambda_n \langle f, e_n \rangle e_n.$$

We also get

$$(A - \frac{1}{2\sqrt{3}}I)^{10}(f) = \Sigma_{n=1}^{\infty}(-i\lambda_n - \frac{1}{2\sqrt{3}})^{10}\langle f, e_n\rangle e_n =$$

$$= (-i\frac{1}{2\sqrt{3}} - \frac{1}{2\sqrt{3}})^{10}\langle f, e_1\rangle e_1 + (i\frac{1}{2\sqrt{3}} - \frac{1}{2\sqrt{3}})^{10}\langle f, e_2\rangle e_2 =$$

$$= (\frac{1}{2\sqrt{3}})^{10}(1+i)^{10}\langle f, e_1\rangle e_1 + (\frac{1}{2\sqrt{3}})^{10}(1-i)^{10}\langle f, e_2\rangle e_2$$

and

$$\|(A - \frac{1}{2\sqrt{3}}I)^{10}(f)\|^2 = (\frac{1}{2\sqrt{3}})^{20}[|1+i|^{20}|\langle f, e_1\rangle|^2 + |1-i|^{20}|\langle f, e_2\rangle|^2] =$$

$$= (\frac{1}{6})^{10}[|\langle f, e_1\rangle|^2 + |\langle f, e_2\rangle|^2].$$

By Parseval´s formula we conclude that

$$\|(A - \frac{1}{2\sqrt{3}}I)^{10}(f)\| \leq (\frac{1}{6})^5\|f\|$$

and also

$$\|(A - \frac{1}{2\sqrt{3}}I)^{10}\| = (\frac{1}{6})^5 (= \frac{1}{7776}).$$

3. Let $g \in L^2([0,1])$ be a fixed function and consider the equation

$$(x+1)\int_0^1 tf(t)\,dt = f(x) + g(x), \quad x \in [0,1]$$

for $f \in L^2([0,1])$. Show that this equation has a unique solution.

(3p)

Solution: For fixed $g \in L^2([0,1])$ consider the equation

$$f(x) = (x+1)\int_0^1 tf(t)\,dt - g(x) \quad x \in [0,1] \tag{1}$$

Set

$$T(f)(x) = \text{RHS } (1)$$

We see that

$$\|T(f)\|_{L^2} \leq |\int_0^1 tf(t)\,dt| \cdot \|x+1\|_{L^2} + \|g\|_{L^2}$$

where

$$|\int_0^1 tf(t)\,dt|^2 \leq (\int_0^1 |t||f(t)|\,dt)^2 \leq \int_0^1 t^2\,dt \cdot \|f\|_{L^2}^2 = \frac{1}{3}\|f\|_{L^2}^2 < \infty$$

and

$$\|x+1\|_{L^2} = (\int_0^2 (x+1)^2\,dx)^{\frac{1}{2}} = \sqrt{\frac{7}{3}} < \infty.$$

Moreover $T : L^2([0,1]) \to L^2([0,1])$ is a contraction since

$$\|T(u) - T(v)\|_{L^2} = \|(x+1)\int_0^1 t(u(t) - v(t))\,dt\|_{L^2} \leq$$

$$\leq \sqrt{\frac{7}{3}} \cdot \sqrt{\frac{1}{3}} \|u - v\|_{L^2} = \sqrt{\frac{7}{9}} \|u - v\|_{L^2}.$$

We can now conclude by Banach´s fixed point theorem that (1) has a unique solution $f \in L^2([0,1])$ since $(L^2([0,1]), \| \cdot \|_{L^2})$ is a Banach space.

4. Let X be a Banach space and assume that $A \in \mathcal{B}(X,X)$ with operator-norm $\|A\| < 1$. Show that $(I + A)^{-1}$ exists as a mapping $X \to X$ and belongs to $\mathcal{B}(X,X)$.
   Moreover define $\sigma(A)$, the spectrum of A, and $\rho(A)$, the resolvent set of A, and show that $\sigma(A)$ is a compact set in C.

(5p)

Solution: See textbook and lecture notes on course homepage

5. Assume that $x_n \rightharpoonup x$ in a Hilbert space $(H, \langle \cdot, \cdot \rangle)$. Show that
   (a) $A \in \mathcal{B}(H)$ implies that $A(x_n) \rightharpoonup A(x)$ in H, and that
   (b) $A \in \mathcal{K}(H)$ implies that $A(x_n) \to A(x)$ in H.

(4p)

Solution: See textbook

6. Let X be a Banach space and A a non-empty subset of a normed space Y. Let

$$T : X \times A \to X$$

be a continuous mapping[1] and assume that there exits a $k \in [0, 1)$ such that

$$\|T(x_1, y) - T(x_2, y)\| \leq k\|x_1 - x_2\| \text{ for all } x_1, x_2 \in X \text{ and } y \in A.$$

Show that for each fixed $y \in A$ the mapping T has a unique fixed point $x_0(y)$ and that

$$x_0 : A \to X$$

is a continuous mapping.

(4p)

[1] This means that $x_n \to x$ in X and $y_n \to y$ in Y implies that $T(x_n, y_n) \to T(x, y)$ in X.

Solution: For each fixed $y \in A$ we obtain by Banach´s fixed point theorem that there exists a unique fixed point $x_0(y)$ for $T(x, y)$.
Assume that $y_n \to y$ in Y where $\{y_n : n = 1, 2, 3, \ldots\} \bigcup \{y\} \subset A$. This gives

$$\|x_0(y_n) - x_0(y)\| = \|T(x_0(y_n), y_n) - T(x_0(y), y)\| \leq$$

$$\leq \|T(x_0(y_n), y_n) - T(x_0(y), y_n)\| + \|T(x_0(y), y_n) - T(x_0(y), y)\| \leq$$
$$\leq k\|x_0(y_n) - x_0(y)\| + \|T(x_0(y), y_n) - T(x_0(y), y)\|$$

which implies that

$$\|x_0(y_n) - x_0(y)\| \leq \frac{1}{1-k}\|T(x_0(y), y_n) - T(x_0(y), y)\|.$$

Since
$$\|T(x_0(y), y_n) - T(x_0(y), y)\| \to 0$$
as $y_n \to y$ in Y it follows that $x_0(y_n) \to x_0(y)$ in X as $y_n \to y$ in Y.