Written exam TMA401 / MMA400 2015-10-28

Solutions

① Show

$$\begin{cases} u''(x) + 2u'(x) + u(x) = \frac{1}{2}\sin^2(u(x)) & , 0 \le x \le 1 \\ u(0) = u(1) = 0 \end{cases}$$

has a unique solution $u \in C^2([0,1])$

solution: Step 1: Calculate the Green's function

$Lu = u'' + 2u' + u$ has a basis $u_1(x) = e^{-x}$, $u_2(x) = xe^{-x}$ for $N(L)$. Set $e(x,t) = a_1(t)e^{-x} + a_2(t)xe^{-x}$. Here

$$\begin{cases} 0 = e(t,t) = a_1(t)e^{-t} + a_2(t)te^{-t} \\ 1 = e_x'(t,t) = -a_1(t)e^{-t} + a_2(t)(e^{-t} - te^{-t}) \end{cases}$$

implies $a_1(t) = -te^{t}$, $a_2(t) = e^{t}$. Now set

$$g(x,t) = e(x,t)\Theta(x-t) + b_1(t)e^{-x} + b_2(t)xe^{-x}.$$

The conditions

$$\begin{cases} 0 = g(0,t) = b_1(t) \\ 0 = g(1,t) = -te^{t-1} + e^{t-1} + b_1(t)e^{-1} + b_2(t)e^{-1} \end{cases}$$

for $t \in (0,1)$ gives $b_1(t) = 0$, $b_2(t) = (t-1)e^{t}$.

Hence the Green's function is given by

$$g(x,t) = \begin{cases} (t-1)xe^{t-x} & 0 \le x < t \le 1 \\ (x-1)te^{t-x} & 0 \le t < x \le 1 \end{cases}$$

Step 2: Prove that the BVP has a unique solution

Set $T(u)(x) = \int_0^1 g(x,t)\frac{1}{2}\sin^2(u(t))\,dt$, $x \in [0,1]$ for $u \in C([0,1])$. Let $(C([0,1]), \|\cdot\|)$ denote the normed space with $\|v\| = \max_{x\in[0,1]} |v(x)|$. This implies that the normed space is a Banach space. Moreover $T: C([0,1]) \to C([0,1])$, actually $T(u) \in C^2([0,1])$ for $u \in C([0,1])$. We try to apply the Banach's fixed point theorem to prove that the BVP has a unique solution. Fix $u, v \in C([0,1])$.

$$|T(u)(x) - T(v)(x)| = \left|\int_0^1 g(x,t)\tfrac{1}{2}\left(\sin^2(u(t)) - \sin^2(t)\right)dt\right| \le$$

$$\le \frac{1}{2}\int_0^1 |g(x,t)|\cdot|\sin^2(u(t)) - \sin^2(v(t))|\,dt.$$

Here $|\sin^2(a) - \sin^2(b)| \le \max_{c\in\mathbb{R}} \left|\frac{d}{dx}\sin^2(x)\right| \cdot |a-b| =$

$$= \max_{c\in\mathbb{R}} \underbrace{|2\sin c\cos c|}_{\sin(2c)}\,|a-b| = |a-b| \quad \text{for } a,b\in\mathbb{R}$$

and we conclude that

$$|T(u)(x) - T(v)(x)| \le \frac{1}{2}\int_0^1 |g(x,t)|\,dt\,\|u-v\|$$

Moreover $g(x,t) \le 0$ for $(x,t) \in 0 \le x,t \le 1$ and so $\int_0^1 |g(x,t)|\,dt = \int_0^1 g(x,t)\cdot(-1)\,dt \equiv j(x)$, where $j'' + 2j' + j = -1$, $j(0) = j(1) = 0$. This implies $j(x) = Ae^{-x} + Bxe^{-x} - 1$ with $A - 1 = 0 = Ae^{-1} + Be^{-1} - 1$ i.e. $A = 1$, $B = e-1$. Moreover

$$j(x) = e^{-x} + (e-1)xe^{-x} - 1 \le e-1 \quad \text{for } 0 \le x \le 1$$

$$j(x) = \int_0^1 |g(x,t)|\,dt \ge 0 \quad \text{for } 0 \le x \le 1.$$

so $\max_{x\in[0,1]} |j(x)| \le e-1$. This gives

$$\|T(u) - T(v)\| \le \frac{e-1}{2}\|u-v\| \quad \text{for all } u,v \in C([0,1])$$

so T is a contraction on $C([0,1])$ since $\frac{e-1}{2} < 1$. Finally we have that the BVP has a unique solution by Banach's fixed point theorem.

(2) $a \in C([0,1])$ and $A: L^2([0,1]) \to L^2([0,1])$ defined by $A(f)(x) = a(x)f(x)$, $x \in [0,1]$. Show A bounded linear operator on $L^2([0,1])$ and calculate $\|A\|$.

Solution: a continuous (complex-valued) function on $[0,1]$ implies that $M \equiv \max_{x\in[0,1]} |a(x)| = |a(x_0)| < \infty$ for some $x_0 \in [0,1]$. This implies that

$$\|A(f)\|_{L^2} = \left(\int_0^1 |a(x)f(x)|^2\,dx\right)^{1/2} \le \left(\int_0^1 M^2|f(x)|^2\,dx\right)^{1/2} =$$

$$= M\|f\|_{L^2} \quad \text{for all } f \in L^2([0,1]).$$

Moreover A is a linear mapping on $L^2([0,1])$ since for $f, g \in L^2([0,1])$, $\alpha, \beta \in \mathbb{C}$

$$A(\alpha f + \beta g)(x) = a(x)(\alpha f + \beta g)(x) = a(x)(\alpha f(x) + \beta g(x)) =$$
$$= \alpha(a(x) f(x)) + \beta(a(x) g(x)) = \alpha A(f)(x) + \beta A(g)(x) =$$
$$= (\alpha A(f) + \beta A(g))(x) \qquad \text{i.e. } A(\alpha f + \beta g) = \alpha A(f) + \beta A(g).$$

We now know that A is a bounded linear operator on $L^2([0,1])$ with $\|A\| \leq M$. Moreover we note that a is continuous and hence $|a(x)|$ is close to M if x is close to x_0. So we consider functions $f \in L^2([0,1])$ with $\|f\|_{L^2} = 1$ that are concentrated around x_0.

Fix $0 < \varepsilon$ $(< |a(x_0)|)$. $a \in C([0,1])$ implies that there exists an intervall I_ε with $x_0 \in I_\varepsilon$ such that

$$|a(x)| > |a(x_0)| - \varepsilon \qquad \text{for } x \in I_\varepsilon.$$

Let δ_ε denote the length of I_ε. Set $f_\varepsilon = \frac{1}{\sqrt{\delta_\varepsilon}} \chi_{I_\varepsilon}$.

Then $\|f_\varepsilon\|_{L^2} = \left(\int_0^1 \left| \frac{1}{\sqrt{\delta_\varepsilon}} \chi_{I_\varepsilon} \right|^2 dx \right)^{1/2} = \left(\int_{I_\varepsilon} \frac{1}{\delta_\varepsilon} dx \right)^{1/2} = 1$

and

$$\|A(f_\varepsilon)\|_{L^2} = \left(\int_0^1 \left| a(x) \frac{1}{\sqrt{\delta_\varepsilon}} \chi_{I_\varepsilon}(x) \right|^2 dx \right)^{1/2} =$$
$$= \left(\int_{I_\varepsilon} |a(x)|^2 \frac{1}{\delta_\varepsilon} dx \right)^{1/2} \geq \left(\int_{I_\varepsilon} (|a(x_0)| - \varepsilon)^2 \frac{1}{\delta_\varepsilon} dx \right)^{1/2} =$$
$$= (|a(x_0)| - \varepsilon) \left(\int_{I_\varepsilon} \frac{1}{\delta_\varepsilon} dx \right)^{1/2} = |a(x_0)| - \varepsilon = M - \varepsilon$$

This gives

$$\|A\| \geq M - \varepsilon \qquad \text{for all } \varepsilon > 0.$$

We have $\|A\| = M$.

③ $(e_n)_{n=1}^\infty$ complete ON-sequence in a Hilbert space E and $(\alpha_n)_{n=1}^\infty$ bounded sequence of complex numbers. Set

$$A(x) = \sum_{n=1}^\infty \alpha_n \langle x, e_n \rangle e_n, \qquad x \in E$$

1) Calculate the eigenvalues for A

2) Give necessary and sufficient conditions on $(\alpha_n)_{n=1}^{\infty}$ for A to be onto

3) Give an example of $(\alpha_n)_{n=1}^{\infty}$ with the property $R(A)$ is a dense proper subspace of E

Solution: 1) $A \in B(E,E)$ according to homework assignment 3.

λ is an eigenvalue for A if there exists $x \neq 0$ s.t. $A(x) = \lambda x$. Since $(e_n)_{n=1}^{\infty}$ complete ON-seq. we have $x = \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n$ for all $x \in E$. We obtain

$$\sum_{n=1}^{\infty} \alpha_n \langle x, e_n \rangle e_n = \sum_{n=1}^{\infty} \lambda \langle x, e_n \rangle e_n .$$

This implies $(\lambda - \alpha_n)\langle x, e_n \rangle = 0$ for all n.

If $\lambda = \alpha_n$ then $A(e_n) = \lambda e_n$ and hence λ is an eigenvalue with eigenvector e_n. If $\lambda \neq \alpha_n$ for all n then $\langle x, e_n \rangle = 0$ for all n and $x = 0$

so λ eigenvalue for A iff $\lambda = \alpha_n$ for some n.

2) Claim A is onto $\Leftrightarrow \inf_n |\alpha_n| > 0$

Proof of $\Leftarrow$: Set $a \equiv \inf_n |\alpha_n| > 0$ and fix $y \in E$.

To show: There exists $x \in E$ s.th $A(x) = y$ i.e.

$$\sum_{n=1}^{\infty} \alpha_n \langle x, e_n \rangle e_n = \sum_{n=1}^{\infty} \langle y, e_n \rangle e_n , \quad \text{i.e}$$

$\alpha_n \langle x, e_n \rangle = \langle y, e_n \rangle$ all n.

But $y \in E$ so $(\langle y, e_n \rangle)_{n=1}^{\infty} \in \ell^2$. Then also $(\langle \frac{1}{\alpha_n} y, e_n \rangle)_{n=1}^{\infty} \in \ell^2$ since $\sum_{n=1}^{\infty} |\langle \frac{1}{\alpha_n} y, e_n \rangle|^2 =$

$$= \sum_{n=1}^{\infty} \frac{1}{|\alpha_n|^2} |\langle y, e_n \rangle|^2 \leq \frac{1}{a^2} \sum_{n=1}^{\infty} |\langle y, e_n \rangle|^2 < \infty .$$

Hence $x = \sum_{n=1}^{\infty} \frac{1}{\alpha_n} \langle y, e_n \rangle e_n$ implies

$$A(x) = \sum_{n=1}^{\infty} \alpha_n \langle \sum_{k=1}^{\infty} \frac{1}{\alpha_k} \langle y, e_k \rangle e_k , e_n \rangle e_n =$$

$$= \sum_{n=1}^{\infty} \alpha_n \frac{1}{\alpha_n} \langle y, e_n \rangle e_n = \sum_{n=1}^{\infty} \langle y, e_n \rangle e_n = y$$

Proof of $\Rightarrow$: Assume $\inf_n |\alpha_n| = 0$. Then there exists a sequence $(\alpha_{n_k})_{k=1}^{\infty}$ ($(n_k)_{k=1}^{\infty}$ strictly increasing) s.th $|\alpha_{n_k}| \leq \frac{1}{k}$, $k = 1, 2, \dots$

set $y = \sum_{k=1}^{\infty} \alpha_{n_k} e_{n_k}$. Then $y \in E$ since $(\alpha_{n_k})_{k=1}^{\infty} \in \ell^2$. Now if $A(x) = y$ for some $x \in E$ then $\alpha_{n_k} \langle x, e_{n_k} \rangle = \alpha_{n_k}$ $k = 1, 2 \ldots$ and hence $\langle x, e_{n_k} \rangle = 1$ for all k. But this implies that $x \notin E$ since $(1)_{k=1}^{\infty} \notin \ell^2$.

3) Example of a sequence $(\alpha_n)_{n=1}^{\infty}$ with $\mathcal{R}(A)$ being a proper dense subspace of E. Take e.g. $\alpha_n = \frac{1}{n}$, $n = 1, 2, \ldots$, which has the property $\inf_n |\alpha_n| = 0$ but $\alpha_n \neq 0$ all n. We observe that

- $\mathcal{R}(A)$ is a subspace of E since A is linear
- $\mathcal{R}(A) \neq E$ since $\inf_n |\alpha_n| = 0$
- $\overline{\mathcal{R}(A)} = E$: (Compare homework assignment 3

  We know that $A^* \in \mathcal{B}(E,E)$ is defined by
  $$A^*(x) = \sum_{n=1}^{\infty} \overline{\alpha_n} \langle x, e_n \rangle e_n \quad , \ x \in E$$
  and that $\overline{\mathcal{R}(A)} = \mathcal{N}(A^*)^{\perp}$. Moreover A^* is 1-1 since $\alpha_n \neq 0$ all n and hence $\mathcal{N}(A^*) = \{0\}$. This gives $E = \{0\}^{\perp} = \overline{\mathcal{R}(A)}$)

  Fix $y \in E$. Then $y = \sum_{n=1}^{\infty} \langle y, e_n \rangle e_n$. For $N = 1, 2, \ldots$ set $x_N = \sum_{n=1}^{N} n \langle y, e_n \rangle e_n \in E$

  We note that
  $$A(x_N) = \sum_{n=1}^{\infty} \frac{1}{n} \langle \sum_{k=1}^{N} k \langle y, e_k \rangle e_k , e_n \rangle e_n =$$
  $$= \sum_{n=1}^{N} \langle y, e_n \rangle e_n \longrightarrow y \text{ in } E \text{ as } N \to \infty.$$
  This implies $\overline{\mathcal{R}(A)} = E$.

4 & 5 See textbook

6 $(X, \|\cdot\|)$ Banach space and $T: X \to X$ a contraction. Also $T_n : X \to X$ $n = 1, 2, \ldots$ and $(x_n)_{n=1}^{\infty}$ is a sequence in X s.th. $T_n(x_n) = x_n$ $n = 1, 2 \ldots$

Assume $\lim_{n \to \infty} \sup_{x \in X} \| T_n(x) - T(x) \| = 0$

show that $(x_n)_{n=1}^{\infty}$ converges in $\mathbb{X}$.

Solution. T is a contraction on a Banach space and hence it has a unique fixed point $\bar{x}$ by the Banach's fixed point theorem.

Claim $x_n \to \bar{x}$ in $\mathbb{X}$.

We observe that

$$\|x_n - \bar{x}\| = \|T_n(x_n) - T(\bar{x})\| \leq$$

$$\leq \|T_n(x_n) - T(x_n)\| + \|T(x_n) - T(\bar{x})\| \leq$$

$$\leq \|T_n(x_n) - T(x_n)\| + c\|x_n - \bar{x}\|$$

where $c<1$ from the contraction property of T.

Hence

$$\|x_n - \bar{x}\| \leq \frac{1}{1-c} \|T_n(x_n) - T(x_n)\| \leq$$

$$\leq \frac{1}{1-c} \sup_{x \in \mathbb{X}} \|T_n(x) - T(x)\| \longrightarrow 0, \quad n \to \infty$$

by the hypothesis of the problem. We have shown that the sequence $(x_n)_{n=1}^{\infty}$ converges to the unique fixed point of T.