Solutions for TMA401/MMA 2016-10-26

(1) Show that

$$(*)\quad \begin{cases} (e^x u'(x))' + \dfrac{|u(x)|}{3+(u(x))^2} = 0 \quad , x \in [0,1] \\ u(0)+u'(0) = u(1) - u'(1) = 0 \end{cases}$$

has a unique solution $u \in C^2([0,1])$

Solution: The differential equations can be written

$$u''(x) + u'(x) = -e^{-x} \frac{|u(x)|}{3+(u(x))^2}.$$

Step 1: Calculate the Green's function for $L = D^2 + D$

and $R_1 u = u(0) + u'(0)$, $R_2 u = u(1) - u'(1)$

$u_1(x) = 1$, $u_2(x) = e^{-x}$ form a basis for $N(L)$

Set $e(x,t) = a_1(t) u_1(x) + a_2(t) u_2(x)$ where

$$\begin{cases} 0 = a_1(t) + a_2(t) e^{-t} & (= e(t,t)) \\ 1 = -a_2(t) e^{-t} & (= e_x'(t,t)) \end{cases}$$

This gives $a_1(t) = 1$, $a_2(t) = -e^t$

Moreover set $g(x,t) = e(x,t)\theta(x-t) + b_1(t) u_1(x) + b_2(t) u_2(x)$

where, for $0 < t < 1$

$$\begin{cases} 0 = R_1 g(\cdot,t) = b_1(t) + b_2(t) - b_2(t) \\ 0 = R_2 g(\cdot,t) = 1 - e^{t-1} + b_1(t) + b_2(t)e^{-1} - (e^{t-1} - b_2(t)e^{-1}) \end{cases}$$

This gives $b_1(t) = 0$, $b_2(t) = e^t - \frac{1}{2}e$

We obtain the Green's function

$$g(x,t) = (1 - e^{t-x})\theta(x-t) + (e^t - \tfrac{1}{2}e)e^{-x}$$

Step 2: For $u \in C([0,1])$ set

$$T(u)(x) = \int_0^1 g(x,t)\left[-e^{-t} \frac{|u(t)|}{3+(u(t))^2}\right] dt \quad , x \in [0,1].$$

Here $T: C([0,1]) \longrightarrow C([0,1])$ and $(C([0,1]), \|\cdot\|)$,

where $\|\cdot\|$ denotes the max-norm, is a Banach space. Banach's fixed point theorem gives that T has a unique fixed point and

Hence (*) has a unique solution $u \in C^2([0,1])$ provided T is a contraction.

Fix $u_1, u_2 \in C([0,1])$. For $x \in [0,1]$ consider

$$|T(u_1)(x) - T(u_2)(x)| = \left| \int_0^1 g(x,t)\left[e^{-t}\left(\frac{|u_1(t)|}{3+(u_1(t))^2} - \frac{|u_2(t)|}{3+(u_2(t))^2}\right)\right]dt\right|$$

$$\le \int_0^1 |g(x,t)| \cdot \left| \frac{|u_1(t)|}{3+(u_1(t))^2} - \frac{|u_2(t)|}{3+(u_2(t))^2}\right| dt .$$

For $a, b \in \mathbb{R}$ consider

$$\left| \frac{|a|}{3+a^2} - \frac{|b|}{3+b^2} \right| \le \begin{cases} \left| \frac{a}{3+a^2} - \frac{b}{3+b^2} \right| & \text{if } a\cdot b \ge 0 \\ \frac{|a|}{3} + \frac{|b|}{3} = \frac{1}{3}|a-b| & \text{if } a \cdot b < 0 \end{cases}$$

For $a \cdot b \ge 0$ the mean value theorem gives

$$\left| \frac{a}{3+a^2} - \frac{b}{3+b^2} \right| = \frac{|3-\xi^2|}{(3+\xi^2)^2} |a-b| \le \frac{1}{3}|a-b| \quad \text{for}$$

some ξ between a and b.

This yields

$$\|T(u_1) - T(u_2)\| \le \max_{x \in [0,1]} \int_0^1 |g(x,t)|\,dt \cdot \frac{1}{3} \|u_1 - u_2\| .$$

Here we observe that

$$0 \le x < t \le 1: \quad |g(x,t)| = |(e^t - \tfrac{1}{2}e)e^{-x}| \le \tfrac{1}{2}e < \tfrac{3}{2}$$

$$0 \le t < x \le 1: \quad |g(x,t)| = |1 - \tfrac{1}{2}e^{1-x}| \le \tfrac{1}{2}e < \tfrac{3}{2}$$

So $\|T(u_1) - T(u_2)\| \le \frac{1}{2}\|u_1 - u_2\|$ and T is a contraction.

(2) $E = \{ x = (x_1, x_2, x_3, \dots) \in \ell^2 : x_k \ne 0$ only for finitely many $k \}$

$\langle \cdot, \cdot \rangle$ denotes the inner product in ℓ^2.

Set $M = \{ x \in E : \sum_{\mu=1}^{\infty} x_\mu = 0 \}$. Determine $\overline{M}$ and $M^\perp$.

Solution: Fix $x \in E$ and $\varepsilon > 0$. Then $x = (x_1, x_2, \dots, x_N, 0, 0, \dots)$

Set $a = -(x_1 + x_2 + \dots + x_N)$. Let $\tilde{N}$ denote a positive integer. Consider the element $x_{\tilde{N}}$ in E where

$$x_{\tilde{N}} = (x_1, x_2, \dots, x_N, \underbrace{\tfrac{a}{\tilde{N}}, \tfrac{a}{\tilde{N}}, \dots, \tfrac{a}{\tilde{N}}}_{\tilde{N} \text{ positions}}, 0, 0, \dots).$$

Then $\|x - x_{\tilde{N}}\|_{\ell^2}^2 = |a|^2 \frac{1}{\tilde{N}}$. Choose $\tilde{N}$ large enough so that $|a|^2 \frac{1}{\tilde{N}} < \varepsilon^2$. Then $\|x - x_{\tilde{N}}\| < \varepsilon$.

Hence M is dense in E, i.e. $\overline{M} = E$.

Since M is dense in E we can conclude $M^\perp = \{0\}$.

This follows from $x \in M^\perp$ implies that

$\langle x, y \rangle = 0$ for all $y \in M$ and since $\overline{M} = E$ we

can take a sequence $y_\mu \in M$ s.th. $y_\mu \to x$ in E.

Hence $\|x\|^2 = \langle x, x \rangle = \lim_{\mu \to \infty} \langle x, y_\mu \rangle = 0$ and so

$x = 0$.

Answer: $\overline{M} = E$, $M^\perp = \{0\}$

(3) $x = (x_1, x_2, x_3, \dots)$ sequence of non-negative real numbers and $1 < p < \infty$. Assume

$\sum_{\mu=1}^{\infty} |x_\mu y_\mu| < \infty$ for all $y = (y_1, y_2, \dots) \in \ell^q$

where $\frac{1}{p} + \frac{1}{q} = 1$. Show $x \in \ell^p$.

Solution: We want to apply Banach-Steinhaus theorem.

Set $T_N : \ell^q \ni (y_1, y_2, \dots) \mapsto \sum_{\mu=1}^{N} x_\mu y_\mu \in \mathbb{R}$, $N = 1, 2, \dots$

We note that for every N we get

$|\sum_{\mu=1}^{N} x_\mu y_\mu| \le \|x_N\|_{\ell^p} \cdot \|y\|_{\ell^q}$, $y \in \ell^q$

where $x_N = (x_1, x_2, \dots, x_N, 0, 0, \dots)$, by Hölders inequality. Moreover T_N are linear mappings that are bounded with $\|T_N\| \le \|x_N\|_{\ell^p}$. If we can prove that $\|T_N\| = \|x_N\|_{\ell^p}$ all N then it follows by BS theorem that $\sup_N \|x_N\|_{\ell^p} = \|x\|_{\ell^p} < \infty$ since $(\ell^q, \|\cdot\|_{\ell^q})$ is a Banach space.

For fixed N set $y_N = \left(\frac{1}{\|x_N\|_{\ell^p}}\right)^{p-1} (x_1^{\frac{p}{q}}, x_2^{\frac{p}{q}}, \dots, x_N^{\frac{p}{q}}, 0, 0, \dots)$.

Then

$$T_N(y_N) = \left(\frac{1}{\|x_N\|_{\ell^p}}\right)^{p-1} \sum_{\mu=1}^{N} \underbrace{x_\mu \cdot x_\mu^{\frac{p}{q}}}_{x_\mu^p} = \|x_N\|_{\ell^p}$$

and

$$\|y_N\|_{\ell^q} = \|x_N\|_{\ell^p}.$$

We are done.

(4) See textbook

(5) a) $T \in \mathcal{B}(\mathbb{X},\mathbb{X})$ where $(\mathbb{X}, \|\cdot\|)$ is a Banach space

Assume there exists $C > 0$ s.th

$\|x\| \leq C\|T(x)\|$ all $x \in \mathbb{X}$.

Show that $\mathcal{R}(T)$ is a closed subspace of $\mathbb{X}$.

Solution: Clearly $\mathcal{R}(T)$ is a subspace of $\mathbb{X}$ since T is linear. To show that $\mathcal{R}(T)$ is closed consider $\mathcal{R}(T) \ni y_n \to y$ in $\mathbb{X}$. That $\mathcal{R}(T)$ is closed follows if we can show that $y \in \mathcal{R}(T)$.

Fix $x_n \in \mathbb{X}$ s.th. $T(x_n) = y_n$ for all n.

Then $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{X}$ since

$$\|x_n - x_m\| \leq C\|T(x_n - x_m)\| = C\|T(x_n) - T(x_m)\| =$$
$$= C\|y_n - y_m\| \to 0 \quad \text{as } n, m \to \infty.$$

$\mathbb{X}$ is a Banach space so $(x_n)_{n=1}^{\infty}$ converges. Call the limit element x. Now

$$y \leftarrow y_n = T(x_n) \to T(x) \quad \text{in } \mathbb{X}$$

since T is continuous (since T is bounded).

Hence $y = T(x)$ and $y \in \mathcal{R}(T)$.

b) straight-forward

(6) $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$, $(Y, \|\cdot\|_Y)$, $(Z, \|\cdot\|_Z)$ Banach spaces. Assume that $K \in \mathcal{K}(\mathbb{X}, Y)$ and that $A \in \mathcal{B}(Y, Z)$ is injective.

Show that for every $\varepsilon > 0$ there exists a real number $C_\varepsilon > 0$ s.th for all x

$$\|K(x)\|_Y \leq \varepsilon\|x\|_{\mathbb{X}} + C_\varepsilon\|AK(x)\|_Z.$$

Solution: Assume that this is not true. Then there exists an $\varepsilon > 0$ and a sequence $(x_n)_{n=1}^{\infty}$ in $\mathbb{X}$ s.th

$$\|K(x_n)\|_Y > \varepsilon\|x_n\|_{\mathbb{X}} + n\|AK(x_n)\|_Z. \quad (*)$$

Clearly $x_n \neq 0$ for all n. Set $\tilde{x}_n = \frac{1}{\|x_n\|_X} x_n$.

K compact implies that there exists a subsequence of $(K(\tilde{x}_n))_{n=1}^{\infty}$, call it still $(K(\tilde{x}_n))_{n=1}^{\infty}$, that converges. Call the limit element $y \in Y$.

Hence

$$\|A(y)\|_Z \longleftarrow \|A(K(\tilde{x}_n))\|_Z \leq \frac{1}{n} \|K(\tilde{x}_n)\|_Y \longrightarrow 0 \qquad n \to \infty$$

We get $A(y) = 0$ and since A is injective we have $y = 0$. But (*) implies that

$$\|K(\tilde{x}_n)\|_Y > \varepsilon + n \|A(K(\tilde{x}_n))\|_Z \qquad \text{all } n$$

and

$$\|K(\tilde{x}_n)\|_Y \longrightarrow 0 \quad , \quad n \to \infty .$$

Contradiction. Conclusion in the problem is proven.