**Institutionen för matematiska vetenskaper**
**Chalmers tekniska högskola och Göteborgs universitet**

**Written exam: Functional Analysis TMA401/MMA400**
**Solutions**
**Date: 2017–01–05**

1. Show that the BVP

$$\begin{cases} u''(x) + u(x) + \lambda \cos(1 + u(x)) = 0, \; x \in [0,1] \\ \\ u(0) = u'(0) = 0 \end{cases}$$

has a unique solution $u \in C^2([0,1])$ for $|\lambda| < \epsilon$, $\epsilon$ small. Give an upper bound on $\epsilon$.

(4p)

*Solution*:

**Green´s function** $g(x,t)$**:** We observe that $u_1(x) = \cos x$ and $u_2(x) = \sin x$ form a basis for $\mathcal{N}(L)$ where $Lu = u'' + u$. Set

$$g(x,t) = \theta(x-t)(a_1(t)u_1(x) + a_2(t)u_2(x)) + b_1(t)u_1(x) + b_2(t)u_2(x)$$

where

$$\begin{cases} a_1(t)\cos t + a_2(t)\sin t = 0 \\ -a_1(t)\sin t + a_2(t)\cos t = 1 \end{cases}$$

and

$$\begin{cases} b_1(t) = 0 \\ b_2(t) = 0 \end{cases}$$

We obtain

$$a_1(t) = -\sin t, \; a_2(t) = \cos t.$$

This gives

$$g(x,t) = \theta(x-t)\sin(x-t), \; 0 \le x, t \le 1.$$

**Unique solution for the BVP:** The problem can be rewritten as

$$u(x) = -\lambda \int_0^1 g(x,t)\cos(1 + u(t))\, dt, \; 0 \le x \le 1.$$

Set

$$T(u)(x) = -\lambda \int_0^1 g(x,t)\cos(1 + u(t))\, dt, \; 0 \le x \le 1,$$

where $u \in C([0,1])$. Clearly $T : C([0,1]) \to C([0,1])$. We assume that $C([0,1])$ is equipped with the max-norm, i.e. $\|f\| = \max_{x\in[0,1]} |f(x)|$, which makes $C([0,1])$

into a Banach space. From Banach´s fixed point theorem it follows that the BVP above has a unique solution $u$ if $T$ is a contraction on $C([0,1])$. For $u, v \in C([0,1])$ we get

$$|T(u)(x) - T(v)(x)| = |\lambda|| \int_0^1 g(x,t)(\cos(1+u(t)) - \cos(1+v(t)))\, dt| \leq$$

$$\leq |\lambda| \int_0^1 |g(x,t)| \cdot |\cos(1+u(t)) - \cos(1+v(t))|\, dt.$$

Apply the mean value theorem to obtain

$$|\cos(1+u(t)) - \cos(1+v(t))| \leq \|u - v\| \ \text{ for } t \in [0,1].$$

This yields

$$\|T(u) - T(v)\| \leq |\lambda| \max_{x\in[0,1]} \int_0^1 |g(x,t)|\, dt \cdot \|u - v\|.$$

We see that $g(x,t) \geq 0$ for all $0 \leq x, t \leq 1$ so

$$j(x) \equiv \int_0^1 |g(x,t)|\, dt = \int_0^1 g(x,t)(+1)\, dt,\ 0 \leq x \leq 1$$

will satisfy $j''(x) + j(x) = 1,\ j(0) = 0,\ j'(0) = 0$. A calculation gives

$$j(x) = 1 - \cos x$$

and

$$0 \leq j(x) \leq 1 - \cos 1,\ 0 \leq x \leq 1.$$

Finally we have proved

$$\|T(u) - T(v)\| \leq |\lambda|(1 - \cos 1)\|u - v\|,\ u, v \in C([0,1])$$

and $T$ is a contraction if

$$|\lambda| < \frac{1}{1 - \cos 1}.$$

2. Let $X$ be a Banach space and $A : X \to X$ a bounded linear operator with $\|A^n\| < 1$ for some positive integer $n$. Show that $I - A$ is a bijection and that its inverse $(I - A)^{-1} : X \to X$ is continuous.

(4p)

*Solution*: Here $A^n \in \mathcal{B}(X)$, since $A \in \mathcal{B}(X)$, and $I - A^n$ is an invertible operator on $X$ by the Neumann-series lemma. In particular, $I - A^n$ is a bijection on $X$. From

$$I - A^n = (I - A)(I + A + \ldots + A^{n-1})$$

we conclude that $I - A$ is a surjection and

$$I - A^n = (I + A + \ldots + A^{n-1})(I - A)$$

implies that $I - A$ is an injection. Hence $I - A$ is a bijection. Moreover from $(I - A^n)^{-1} \in \mathcal{B}(X)$ we obtain

$$I = (I - A)(I + A + \ldots + A^{n-1})(I - A^n)^{-1} = (I + A + \ldots + A^{n-1})(I - A^n)^{-1}(I - A)$$

since

$$(I - A^n)^{-1}(I - A) = (I - A)(I - A^n)^{-1},$$

which follows from $(I - A^n)^{-1} = \Sigma_{k=0}^{\infty} A^{kn}$, and so

$$(I - A)^{-1} = (I + A + \ldots + A^{n-1})(I - A^n)^{-1} \in \mathcal{B}(X).$$

This gives that $(I - A)^{-1}$ is a bounded linear operator on $X$ and hence continuous.

3. Let $E$ be a Hilbert space and $A : E \to E$ a bounded linear and self-adjoint operator that satisfies $A^3 = A^2$. Show that $A$ is an orthogonal projection operator.

(4p)

*Solution*: A bounded linear operator $A$ on a Hilbert space $E$ is an orthogonal projection operator if and only if $A$ is self-adjoint and $A^2 = A$. Hence it remains to show that $A^2 = A$. From $A^3 = A^2$ it follows that $A^4 = A^3 = A^2$ and hence

$$(A^2 - A)^2 = A^4 - 2A^3 + A^2 = \mathbb{0}.$$

Set $T = A^2 - A$. We conclude that $T^* = T$, since

$$T^* = (A^2 - A)^* = (A^2)^* - A^* = (A^*)^2 - A^* = A^2 - A = T,$$

and hence $T^*T = \mathbb{0}$. But this implies that

$$0 = \langle T^*T(x), x\rangle = \langle T(x), T^{**}(x)\rangle = \langle T(x), T(x)\rangle = \|T(x)\|^2$$

for all $x \in E$. Hence $T(x) = \mathbb{0}$ for all $x \in E$ which gives $T = \mathbb{0}$. This shows $A^2 = A$.

4. Let $E$ be a Hilbert space and $(x_n)_{n=1}^{\infty}$ a weakly converging sequence in $E$. Show that

$$\sup_{n=1,2,3,\ldots} \|x_n\| < \infty.$$

Give an example where $(x_n)_{n=1}^{\infty}$ converges weakly to $x$ but

$$\|x_n\| \not\to \|x\| \text{ as } n \to \infty.$$

What can be said if $(x_n)_{n=1}^{\infty}$ converges weakly to $x$ and

$$\|x_n\| \to \|x\| \text{ as } n \to \infty?$$

(5p)

*Solution*: See textbook and lecture notes on course webpage.

5. Let $\mathcal{P}_n$ denote the vector space of all polynomials of degree at most $n$ on $\mathbb{R}$, where $n$ is a positive integer. Set
$$\|p\| = \Sigma_{k=0}^{n}|p(k)|, \quad p \in \mathcal{P}_n.$$
Show that $(\mathcal{P}_n, \|\cdot\|)$ is a Banach space.

(4p)

*Solution*: A sketch: Clearly $\mathcal{P}_n$ is a vector space since it is a function space on $\mathbb{R}$ so it remains to show that
$$\|p\| = \Sigma_{k=0}^{n}|p(k)|, \quad p \in \mathcal{P}_n$$
defines a norm on $\mathcal{P}_n$ and that this is complete. To show that $\|p\| = 0$ implies $p = \mathbb{0}$ we observe that a non-trivial polynomial of degree at most $n$ can at most have $n$ distinct zeros. The other norm axioms are easy to verify for $\|\cdot\|$. To show that $(\mathcal{P}_n, \|\cdot\|)$ is a Banach space we observe that $\mathcal{P}_n$ is a finite-dimensional vector space ($\dim(\mathcal{P}_n) = n+1$) and that all norms for finite-dimensional vector spaces are equivalent. For instance $|||p||| = \Sigma_{k=0}^{n}|a_k|$, where $p(x) = \Sigma_{k=0}^{n}a_k x^k$, defines a another norm for $\mathcal{P}_n$ and it easy to show with this norm that every Cauchy sequence in $\mathcal{P}_n$ converges.

6. Let $E$ be a separable Hilbert space and $A : E \to E$ a linear compact operator. Show that for every $\epsilon > 0$ there exists a finite-rank operator $B$ such that $\|A - B\| < \epsilon$.

(4p)

*Solution*: A sketch: Let $(x_n)_{n=1}^{\infty}$ be an ON-basis for $E$ and for $N = 1, 2, 3, \ldots$ define
$$A_N(x) = A(\Sigma_{n=1}^{N}\langle x, x_n\rangle x_n) = \Sigma_{n=1}^{N}\langle x, x_n\rangle A(x_n), \quad x \in E.$$
Then $A_N$ defines a sequence of linear finite-rank operators on $E$ ($\dim \mathcal{R}(A_N) \leq N$ for each $N$) such that
$$\|A - A_N\| \to 0 \text{ as } N \to \infty.$$
Here it is crucial that $A$ is a compact operator and its convergence improving property is used (every weakly converging sequence in $E$ is mapped by $A$ onto a strongly converging sequence in $E$).