Lösningsförslag till TMA401/MMA400 - tenta 2017-08-26

① Show $5u'' + \frac{1}{1+u^4(x)} = 1$, $x \in [0,1]$, $u(0) = u(1) = 0$ has a unique solution $u \in C^2([0,1])$

Solution: Step 1 Calculate the Green's function $g(x,t)$ for $Lu = u''$, $u(0) = u(1) = 0$. We have

$$g(x,t) = (a_1(t)\cdot 1 + a_2(t)\cdot x)\Theta(x-t) + b_1(t)\cdot 1 + b_2(t)\cdot x$$

where

$$\begin{cases} a_1(t) + a_2(t)\cdot t = 0 \\ a_2(t) = 1 \end{cases} \quad \text{i.e.} \quad \begin{cases} a_1(t) = -t \\ a_2(t) = 1 \end{cases}$$

and

$$\begin{cases} g(0,t) = 0 \\ g(1,t) = 0 \end{cases} \quad t \in (0,1) \quad \text{i.e.} \quad \begin{cases} b_1(t) = 0 \\ b_2(t) = t-1 \end{cases}$$

which gives

$$g(x,t) = \begin{cases} (t-1)x & 0 \le x \le t \le 1 \\ (x-1)t & 0 \le t < x \le 1 \end{cases}$$

Step 2: Set

$$T(u)(x) = \int_0^1 g(x,t)\left(\frac{1}{5} - \frac{1}{5}\frac{1}{1+u^4(t)}\right)dt, \quad u \in C([0,1]).$$

If $T: C([0,1]) \to C([0,1])$ is a contraction then the BVP has a unique solution $u \in C^2([0,1])$ by Banach's fixed point theorem. Fix $u, v \in C([0,1])$.

$$|T(u)(x) - T(v)(x)| \le \frac{1}{5}\int_0^1 |g(x,t)|\left|\frac{1}{1+u^4(t)} - \frac{1}{1+v^4(t)}\right|dt \le$$

$$\le \{\text{mean value thm}\} \le \frac{4}{5}\int_0^1 |g(x,t)|\,dt\, \|u-v\|_\infty \le$$

$$\le \frac{4}{5}\|u-v\|_\infty.$$

Hence $\|T(u) - T(v)\|_\infty \le \frac{4}{5}\|u-v\|_\infty$ where $\|\cdot\|_\infty$ is the max-norm (so $(C([0,1]), \|\cdot\|_\infty)$ is Banach space.).

② Set $Tf(x) = \int_{-\infty}^{\infty} \frac{1}{1+(x-y)^2} f(y)\,dy$, $f \in L^2(\mathbb{R})$. Prove that T is bounded linear self-adjoint operator on $L^2(\mathbb{R})$. Also show that T is not compact.

Solution: Linearity easy (so omitted).

Boundedness: For $f \in L^2(\mathbb{R})$

$$\int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} \frac{1}{1+(x-y)^2} f(y)\, dy \right|^2 dx \leq \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{1+(x-y)^2}} \cdot \frac{|f(y)|}{\sqrt{1+(x-y)^2}} dy \right)^2 dx \leq$$

$$\leq \{\text{Hölder}\} \leq \int_{-\infty}^{\infty} \Big(\underbrace{\int_{-\infty}^{\infty} \frac{1}{1+(x-y)^2} dy}_{x\text{-independent}} \cdot \int_{-\infty}^{\infty} \frac{1}{1+(x-y)^2} |f(y)|^2 dy \Big) dx \leq$$

$$\leq \{\text{Fubini}\} \leq \int_{-\infty}^{\infty} \frac{1}{1+y^2} dy \cdot \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \frac{1}{1+(x-y)^2} dx \, |f(y)|^2 \right) dy =$$

$$= \left(\int_{-\infty}^{\infty} \frac{1}{1+y^2} dy \right)^2 \cdot \|f\|_{L^2}^2 = \pi^2 \cdot \|f\|_{L^2}^2$$

Hence $\|Tf\|_{L^2} \leq \pi \|f\|_{L^2}$ all $f \in L^2(\mathbb{R})$

Self-adjointness: Note that $Tf(x) = \int_{-\infty}^{\infty} k(x,y) f(y)\, dy$

where $k(x,y) = \overline{k(y,x)}$ all $(x,y) \in \mathbb{R}^2$. This implies $T^* = T$.

T not compact: Set $f_n = \chi_{[n,n+1]}$, $n = 1, 2, \ldots$

Then $|\langle f_n, g \rangle_{L^2}| = \left| \int_n^{n+1} g(y)\, dy \right| \leq \int_n^{n+1} |g(y)|\, dy \leq$

$\leq \{\text{Hölder}\} \leq \left(\int_n^{n+1} |g(y)|^2 dy \right)^{1/2} \to 0, \; n \to \infty$

for all $g \in L^2(\mathbb{R})$.

Hence $f_n \rightharpoonup 0$ in $L^2(\mathbb{R})$. But

$$\|Tf_n\|_{L^2}^2 = \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} \frac{1}{1+(x-y)^2} f_n(y)\, dy \right|^2 dx =$$

$$= \int_{-\infty}^{\infty} \left(\int_n^{n+1} \frac{1}{1+(x-y)^2} dy \right)^2 dx =$$

$$= \int_{-\infty}^{\infty} \Big[\arctan(y-x) \Big]_n^{n+1} dx =$$

$$= \int_{-\infty}^{\infty} \arctan(n+1-x) - \arctan(n-x)\, dx =$$

$$= \int_{-\infty}^{\infty} \big(\arctan(1-x) - \arctan(-x) \big) dx = \text{constant} > 0$$

So

$$Tf_n \not\to 0 \text{ in } L^2(\mathbb{R})$$

Hence T is not compact.

③ Set $L(f) = \sum_{n=1}^{\infty} c_n f(x_n)$ where $(c_n)_{n=1}^{\infty} \in \ell^1$, $f \in C(\mathbb{R})$

$(x_n)_{n=1}^{\infty} \subset \mathbb{R}$ with $x_n \neq x_m$ for $n \neq m$ and with no cluster point. Calculate $\|L\|$

Solution: Clearly for $f \in C(\mathbb{R})$ with $|f| \le 1$ we have

$$|L(f)| \le \sum_{n=1}^{\infty} |c_n| |f(x_n)| \le \sum_{n=1}^{\infty} |c_n|$$

Fix arbitrary positive integer N. Since $\min_{\substack{n \ne m \\ n \in \{1,2,\dots,N\},\, m \in \mathbb{Z}_+}} |x_n - x_m| \equiv \varepsilon > 0$

we can construct a $f_N \in C(\mathbb{R})$ such that

$\|f_N\| = \sup_{x \in \mathbb{R}} |f_N(x)| = 1$ and $f_N(x_n) = \frac{\bar{c}_n}{|c_n|}$, $n = 1, 2, \dots, N$

($f(x_n) = 0$ if $c_n = 0$), and $f(x_k) = 0$, $k = N+1, N+2, \dots$

Then $L(f_N) = \sum_{n=1}^{N} |c_n|$. Hence $\|L\| \ge \sup_N \sum_{n=1}^{N} |c_n| = \sum_{n=1}^{\infty} |c_n|$

④ See textbook

⑤ $T \ne 0$ linear functional on normed space X. Prove that (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c), where

a) T continuous

b) $N(T)$ proper closed subspace of X

c) $N(T)$ not dense in X

Proof: a) $\Rightarrow$ b) : $N(T)$ closed subspace since T continuous (easy)

$N(T)$ proper since $T \ne 0$

b) $\Rightarrow$ c) : Since $\overline{N(T)} = N(T)$ is a proper subspace of X it is not dense

c) $\Rightarrow$ a) : We show that T not continuous implies $N(T)$ dense in X. If T not continuous lin. functional there exists a sequence $(x_n)_{n=1}^{\infty}$ in X s.th. $\|x_n\| = 1$ and $|T(x_n)| > n$ $n = 1, 2, \dots$ Fix $y \in X$. Set

$$y_n = y - \frac{T(y)}{T(x_n)} x_n \quad n = 1, 2, \dots$$

Then $T(y_n) = T(y) - T(y) = 0$ so $y_n \in N(T)$ and $\|y_n - y\| = \frac{|T(y)|}{|T(x_n)|} \|x_n\| =$

$$= \frac{|T(y)|}{n} \to 0 \quad , n \to \infty$$

Hence $N(T)$ is dense in X. $\boxed{QED}$

⑥ Let M be a dense subspace of a separable Hilbert space H. Prove that H has an ON-basis consisting of elements in M.

Proof: If H is finite-dimensional the only dense subspace of H is H itself so the result is trivial. Suppose that H is infinite dimensional and let (x_n) denote a countable dense subset of H.

M dense in H implies that for every x_n there exists a sequence $(x_{m,n})_{m=1}^{\infty}$ in M s.th. $x_{m,n} \to x_n$ in $(H, \|\cdot\|)$ as $m \to \infty$. Set $A = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \{x_{m,n}\}$. Here A is a countable subset of M that is dense in H. Set $A = \{z_k : k \in \mathbb{Z}_+\}$.

Let $\tilde{A}$ be lin. indep. subset of A s.th. the set of all finite lin. comb. of $\tilde{A}$ = set of all finite lin. comb. of A. Apply the Gram-Schmidt orthogonalization procedure for $\tilde{A} \equiv \{\tilde{z}_k : k = 1, 2, \ldots\}$. This gives an ON-sequence $(e_k)_{k=1}^{\infty}$ where each e_k is a finite lin. comb. of elements $\tilde{z}_n$ in M and hence $e_k \in M$.

Moreover the closed span of $\{e_k : k = 1, 2, \ldots\}$ is equal to that of A and hence $(e_k)_{k=1}^{\infty}$ is an ON-basis for H.