Solutions to TMA401/MMA given 2017-10-25

(1) $u'' + u' + \lambda |u| = f \in C([0,1])$ with $u(0) = u(1) = 0$ and $|\lambda| < e(e-1)$

Show that this BVP has a unique solution $u \in C^2([0,1])$

solution: <u>Step 1</u>. Set $Lu = u'' + u'$ and $R_1 u = u(0)$, $R_2 u = u(1)$ and calculate the corresponding Green's function $g(x,t)$

Here $u_1(x) = 1$, $u_2(x) = e^{-x}$ is a basis for $N(L)$. Set

$$g(x,t) = \theta(x-t)\underbrace{(a_1(t)u_1(x) + a_2(t)u_2(x))}_{= e(x,t)} + b_1(t)u_1(x) + b_2(t)u_2(x)$$

where

- $e(t,t) = 0$, $e_x'(t,t) = 1$ $\quad t \in [0,1]$
- $R_1 g(\cdot,t) = R_2 g(\cdot,t) = 0$ $\quad t \in (0,1)$

This gives

$$\begin{cases} a_1(t) + a_2(t)e^{-t} = 0, \ -a_2(t)e^{-t} = 1 & t \in [0,1] \\ b_1(t) + b_2(t) = a_1(t) + a_2(t)e^{-1} + b_1(t) + b_2(t)e^{-1} = 0, & t \in (0,1) \end{cases}$$

and hence

$$\begin{cases} a_1(t) = 1, \quad a_2(t) = -e^{t} \\ b_1(t) = \frac{e^t - e}{e-1}, \quad b_2(t) = \frac{e - e^t}{e-1} \end{cases}.$$

We have $g(x,t) = \theta(x-t)(1 - e^{t-x}) + \frac{e^t - e}{e-1} + \frac{e-e^t}{e-1} e^{-x}$

for $x, t \in [0,1]$. We observe that $g(x,t) \leq 0$ for all x,t.

<u>Step 2</u>: Set $T(u)(x) = \int_0^1 g(x,t)(f(t) - \lambda|u(t)|)\,dt$, $x \in [0,1]$

for $u \in C([0,1])$. Then the BVP has a unique solution $u \in C^2([0,1])$ if and only if $T: C([0,1]) \to C([0,1])$ has a unique fixed point. We use the max-norm for $C([0,1])$, i.e. $\|h\| = \max_{x \in [0,1]} |h(x)|$, so $(C([0,1]), \|\cdot\|)$ becomes a Banach space. Banach's fixed point theorem implies that T has a unique fixpoint provided T is a contraction $C([0,1])$. For $u, v \in C([0,1])$

$$|T(u)(x) - T(v)(x)| = \left|\int_0^1 g(x,t)(\lambda|v(t)| - \lambda|u(t)|)\,dt\right| \leq$$

$\leq |\lambda| \int_0^1 |g(x,t)| \underbrace{\big||u(t)| - |v(t)|\big|}_{\leq |u(t)-v(t)|} dt \leq |\lambda| \int_0^1 |g(x,t)| \, dt \, \|u-v\|$

This implies $\quad \|T(u) - T(v)\| \leq |\lambda| \int_0^1 |g(x,t)| \, dt \, \|u-v\|$.

To estimate $\int_0^1 |g(x,t)| dt$ we observe that

$\int_0^1 |g(x,t)| dt = \int_0^1 g(x,t) \cdot (-1) dt = j(x)$

where $j'' + j' = -1$, $j(0) = j(1) = 0$.

Here $j(x) = -x + a + b e^{-x}$ where $j(0) = j(1) = 0$ i.e.

$j(x) = -x + \frac{e}{e-1} - \frac{e}{e-1} e^{-x}$. Straight forward calculations

give $\max_{x \in [0,1]} j(x) = j(\ln \frac{e}{e-1}) = \ldots = \frac{1}{e-1} + \ln(1 - \frac{1}{e}) \leq$

$\leq \frac{1}{e-1} - \frac{1}{e} = \frac{1}{e(e-1)}$. This implies that T is

a contraction on $C([0,1])$ if $\frac{|\lambda|}{e(e-1)} < 1$, and so

by the assumption on $|\lambda|$ in the problem, we obtained

the desired result.

(2) $A : C([0,1]) \to C([0,1])$ where $A(f)(x) = \int_0^x f(y) dy - \int_x^1 f(y) dy$

for $x \in [0,1]$ and $f \in C([0,1])$. Show that $A \in \mathcal{B}(E, E)$

with $E = (C([0,1]), \|\cdot\|)$, where $\|\cdot\|$ is the max-norm,

and calculate $\|A\|$.

Solution: Clearly $A(f) \in C([0,1])$ for $f \in C([0,1])$. The

linearity of A is easy to prove and omitted here.

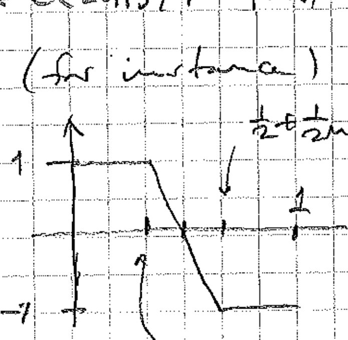
For $f \in C([0,1])$ we have (for $x \in [0,1]$)

$|A(f)(x)| \leq \int_0^x |f(y)| dy + \int_x^1 |f(y)| dy = \int_0^1 |f(y)| dy \leq \|f\|$

and hence $\|A(f)\| \leq \|f\|$ for all $f \in C([0,1])$. This

implies $\|A\| \leq 1$. To show $\|A\| \geq 1$ set (for instance)

$$f_n(x) = \begin{cases} 1 & x \in [0, \frac{1}{2} - \frac{1}{2n}] \\ \text{linear} & \text{in } [\frac{1}{2} - \frac{1}{2n}, \frac{1}{2} + \frac{1}{2n}] \\ -1 & x \in [\frac{1}{2} + \frac{1}{2n}, 1] \end{cases}$$



for $n = 1, 2, \ldots$. Then

$\|f_n\| = 1$ for $n = 1, 2, \ldots$ and

$A(f_n)(\frac{1}{2}) \geq 1 - \frac{1}{n}$ $\quad n = 1, 2, \ldots$

This implies $\|A(f_n)\| \geq 1 - \frac{1}{n}$, $n \in 1, 2, \ldots$

and so $\|A\| \geq \sup_n (1 - \frac{1}{n}) = 1$. This gives $\|A\| = 1$.

(3) $k \in C([0,1] \times [0,1])$ with $k(x,y) > 0$ all $(x,y) \in [0,1] \times [0,1]$.

$A(f)(x) = \int_0^1 k(x,y) f(y)\, dy$, $x \in [0,1]$ and $f \in C([0,1])$.

Show that $A : C([0,1]) \to C([0,1])$ has a positive eigenvalue with an eigenfunction $f \in C([0,1])$ with $f(x) > 0$ all $x \in [0,1]$.

Solution: Set

$$F = \{ f \in C([0,1]) : \|f\| = 1,\ f \geq 0 \}$$

Then F is a closed convex set in the Banach space $(C([0,1]), \|\cdot\|)$, where $\|\cdot\|$ denotes the max-norm.

We observe that $\|A(f)\| > 0$ for $f \in F$ since $k > 0$ on $[0,1] \times [0,1]$ and $f \geq 0$ with $f(x) > 0$ for some $x \in [0,1]$. Now set

$$T(f) = \frac{1}{\|A(f)\|} A(f) \quad \text{for } f \in F.$$

Then

- $T : F \to F$ continuous (easy to show)
- $T(F)$ relatively compact in $C([0,1])$ by Arzela-Ascoli theorem (also easy to show)

Schauder's fixed point theorem (generalized version) implies that T has a fixed point $\tilde{f} \in F$.

Hence

$$A(\tilde{f}) = \|A(\tilde{f})\| \cdot \tilde{f}$$

where $\tilde{f}(x) > 0$ all $x \in [0,1]$ since $k > 0$ on $[0,1] \times [0,1]$.

(4) See textbook.

(5) a) See textbook

b) E Hilbert space, $A \in \mathcal{B}(E,E)$ where

$A^*A \in \mathcal{K}(E,E)$. Show $A \in \mathcal{K}(E,E)$

Proof: From the textbook we know that

$A \in \mathcal{K}(E,E)$ if and only if $x_n \rightharpoonup x$ in E implies

$A(x_n) \to A(x)$ in E. Since A is linear

the last statement is equivalent to

$x_n \rightharpoonup 0$ in $E \Rightarrow A(x_n) \to 0$ in E.

Assume $x_n \rightharpoonup 0$ in E. This implies

$\sup_n \|x_n\| < \infty$ by well-known theorem.

Moreover

$$\|A(x_n)\|^2 = \langle A(x_n), A(x_n)\rangle = \langle A^*A(x_n), x_n\rangle \le$$

$$\le \|A^*A(x_n)\| \cdot \|x_n\| \le \|A^*A(x_n)\| \cdot \sup_n \|x_n\|$$

Here $A^*A \in \mathcal{K}(E,E)$ and $x_n \rightharpoonup 0$ in E so

$A^*A(x_n) \to 0$ in E, i.e. $\|A^*A(x_n)\| \to 0$, $n \to \infty$

We obtain $\|A(x_n)\| \to 0$ $n \to \infty$ and so

$A(x_n) \to 0$ in E. $\boxed{}$

(6) $(x_n)_{n=1}^{\infty}$ bounded sequence in a Hilbert space E.

Show that $(x_n)_{n=1}^{\infty}$ has a weakly convergent subsequence

Proof: Set $S = \overline{\text{Span}\{x_n : n = 1, 2, \ldots\}}$.

Then S is a closed subspace of E and $E = S \oplus S^{\perp}$ by

OPT. Clearly $\langle x_n, y\rangle = 0$ for all n for $y \in S^{\perp}$.

By Gram-Schmidt orthogonalization procedure we

obtain an ON-basis $(e_n)_{n=1}^{N}$ for S. Here N

is finite or infinite. Assume $N = \infty$. (N finite easy)

$(\langle x_n, e_1\rangle)_{n=1}^{\infty}$ is a bounded sequence in $\mathbb{C}$ and hence has a convergent subsequence $(\langle x_{n,1}, e_1\rangle)_{n=1}^{\infty}$. $(\langle x_{n,1}, e_2\rangle)_{n=1}^{\infty}$ bounded in $\mathbb{C}$ and hence has a convergent subsequence $(\langle x_{n,2}, e_2\rangle)_{n=1}^{\infty}$. Proceed inductively. For every positive integer k we obtain a subsequence $(x_{n,k})_{n=1}^{\infty}$ of $(x_{n,k-1})_{n=1}^{\infty}$ and hence of the original sequence $(x_n)_{n=1}^{\infty}$ s.th $(\langle x_{n,k}, e_\ell\rangle)_{n=1}^{\infty}$ converges in $\mathbb{C}$ for $\ell = 1, 2, \dots, k$. Consider the diagonal sequence $(y_n)_{n=1}^{\infty}$, where $y_n = x_{n,n}$.

Here $(\langle y_n, e_\ell\rangle)_{n=1}^{\infty}$ converges for $\ell = 1, 2, \dots$.

Fix $z \in S$. Then $z = \sum_{\ell=1}^{\infty} \langle z, e_\ell\rangle e_\ell$. Here $\sum_{\ell=1}^{\infty} |\langle z, e_\ell\rangle|^2 < \infty$ by Bessel's inequality, and $z_k \to z$ in E where $z_k = \sum_{\ell=1}^{k} \langle z, e_\ell\rangle e_\ell$.

Claim: $(\langle y_n, z\rangle)_{n=1}^{\infty}$ converges in $\mathbb{C}$ (and hence also $(\langle z, y_n\rangle)_{n=1}^{\infty}$)

Fix $\varepsilon > 0$. Set $M = \sup_n \|x_n\|$. Fix k positive integer s.th $\|z - z_k\| < \frac{\varepsilon}{3M}$. Then

$$|\langle z, y_n\rangle - \langle z, y_m\rangle| \le |\langle z - z_k, y_n\rangle| + |\langle z_k, y_n - y_m\rangle| +$$
$$+ |\langle z - z_k, y_m\rangle| \le \|z - z_k\| \cdot \|y_n\| + |\langle z_k, y_n - y_m\rangle| +$$
$$+ \|z - z_k\| \cdot \|y_m\| < \frac{2}{3}\varepsilon + |\langle z_k, y_n - y_m\rangle|.$$

Fix N s.th $|\langle z_k, y_n\rangle - \langle z_k, y_m\rangle| < \frac{\varepsilon}{3}$ for all $n, m \ge N$.

Then $|\langle z, y_n\rangle - \langle z, y_m\rangle| < \varepsilon$ for all $n, m \ge N$

$(\mathbb{C}, |\cdot|)$ is a Banach space so $(\langle z, y_n\rangle)_{n=1}^{\infty}$ converges.

Call the limit $f(z)$. We have $f: E \to \mathbb{C}$ that is linear, since $\langle \alpha_1 z_1 + \alpha_2 z_2, y_n\rangle = \alpha_1\langle z_1, y_n\rangle + \alpha_2\langle z_2, y_n\rangle$ which implies $f(\alpha_1 z_1 + \alpha_2 z_2) = \alpha_1 f(z_1) + \alpha_2 f(z_2)$, and f bounded, since $|f(z)| = |\lim_{n\to\infty} \underbrace{\langle z, y_n\rangle}_{|\cdot| \le \|z\|\cdot\|y_n\| \le \|z\|\cdot M}| \le M \cdot \|z\|$.

Riesz repr theorem implies that

$f(z) = \langle z, y_f\rangle$, $z \in E$, for a uniquely defined $y_f \in E$.

We have that

$$\langle y_n, z\rangle \longrightarrow \langle y_f, z\rangle \ , \ n\to\infty \quad \text{in } (\mathbb{C}, |\cdot|)$$

for all $z \in S$. We have that $y_n \longrightarrow y_f$ in E, note that $y_f \in S$, and $(y_n)_{n=1}^{\infty}$ is a subsequence of $(x_n)_{n=1}^{\infty}$ ⊠