Solutions to TMA401 / MMA400 , 2018-01-04

① $f \in C([0,1])$, $|\lambda| < 2$

$$(*)\begin{cases} u'' - \lambda\sqrt{1+|u|^2} = f(x) , & x \in [0,1] \\ u(0) = u(1) = 0 \end{cases}$$

Show (*) has a unique solution $u \in C^2([0,1])$

Solution: Step 1 : Calculate the Green's function $g(x,t)$ to $Lu = u''$, $Ru = (u(0), u(1)) = (0,0)$,

$u_1(x) = 1$, $u_2(x) = x$ is a basis for $N(L)$. Set

$g(x,t) = (a_1(t)u_1(x) + a_2(t)u_2(x))\,\theta(x-t) + b_1(t)u_1(x) + b_2(t)u_2(x)$

where

$$\begin{cases} a_1(t) + a_2(t)t = 0 \\ a_2(t) = 1 \end{cases} \quad \text{i.e.} \quad \begin{cases} a_1(t) = -t \\ a_2(t) = 1 \end{cases}$$

$$\begin{cases} b_1(t) = 0 \\ -t + 1 + b_1(t) + b_2(t) = 0 \end{cases} \quad \text{i.e.} \quad \begin{matrix} b_1(t) = 0 \\ b_2(t) = t-1 \end{matrix}$$

This yields

$$g(x,t) = (x-t)\theta(x-t) + (t-1)x =$$
$$= \begin{cases} (t-1)x & 0 \le x < t \le 1 \\ (x-1)t & 0 \le t < x \le 1 \end{cases}$$

Step 2 : Fixed point argument.

Set $T(u)(x) = \int_0^1 g(x,t)\left[f(t) + \lambda\sqrt{1+|u(t)|^2}\right]dt$

for $u \in C([0,1])$. $(C([0,1]), \|\cdot\|)$ with $\|g\| = \max\limits_{x\in[0,1]} |g(x)|$ is a Banach space. If T is a contraction on $C([0,1])$ then Banach's fixed point theorem implies that T has a unique fixed point $u \in C([0,1])$ and hence (*) has a unique solution $u \in C^2([0,1])$. Fix $u, v \in C([0,1])$

$$|T(u)(x) - T(v)(x)| = \left|\int_0^1 g(x,t)\lambda\left(\sqrt{1+|u(t)|^2} - \sqrt{1+|v(t)|^2}\right)dt\right| \le$$
$$\le |\lambda|\int_0^1 |g(x,t)|\frac{|u(t)+v(t)|\,|u(t)-v(t)|}{\sqrt{1+|u(t)|^2} + \sqrt{1+|v(t)|^2}}\,dt \le$$

$\leq |\lambda| \int_0^1 |g(x,t)|\,dt \,\|u-v\|$

Here $g(x,t) \leq 0$ and $j(x) \equiv \int_0^1 -g(x,t)\,dt$ satisfies

$j''(x) = -1$, $j(0) = j(1) = 0$, i.e. $j(x) = \frac{1}{2}(x - x^2)$.

Hence $\max_{x\in[0,1]} j(x) = \frac{1}{8}$

We conclude that $\|T(u) - T(v)\| \leq \frac{|\lambda|}{8} \|u-v\|$,

where $|\lambda| < 8$, and hence T is a contraction

(2) $T: C([0,1]) \to C([0,1])$ where $T(f)(x) = \int_0^x f(t)\,dt$

Show:

1) T is not a contraction i.e. there exists no $\lambda \in [0,1)$

s.th $\|T(f) - T(g)\| \leq \lambda \|f-g\|$ for all $f,g \in C([0,1])$

Sol: set $f = 1$ and $g = 0$. Then $\|f\| = 1$ and

$\|T(f)\| = \max_{x\in[0,1]} |\int_0^x 1\,dt| = \max_{x\in[0,1]} x = 1$.

We have $\|T(f) - T(g)\| = \|f - g\| > 0$.

2) T has a unique fixed point

Sol: For $f \in C([0,1])$ we have $T(f) \in C^1([0,1])$. Hence if f is a fixed point then $f' = f$ and $f(0) = T(f)(0) = 0$. This implies that $f(x) = ae^x$ where $a = 0$. Clearly $f = 0$ is a fixed point and also unique.

3) T^2 is a contraction on $C([0,1])$

Sol: For $f \in C([0,1])$,

$$T^2(f)(x) = T(T(f))(x) = \int_0^x (\int_0^t f(s)\,ds\, dt =$$

$$= \int_0^x (\int_s^x dt) f(s)\,ds = \int_0^x (x-s) f(s)\,ds$$

$$\|T^2(f)\| = \max_{x\in[0,1]} |\int_0^x (x-s) f(s)\,ds| \leq$$

$$\leq \max_{x\in[0,1]} \int_0^x (x-s)\,ds \,\|f\| = \max_{x\in[0,1]} \frac{x^2}{2} \|f\| =$$

$$= \frac{1}{2}\|f\|.$$

We conclude that $\|T^2\| \leq \frac{1}{2}$ and

$\|T^2(f) - T^2(g)\| \leq \frac{1}{2} \|f-g\|$ since T linear

4) $f \in C([0,1])$. Claim $T^n f \to 0$ in $C([0,1])$.

Sol: T has the unique fixed point 0 and also T^2 has the unique fixed point 0 since T^2 is a contraction. From the generalised Banach's fixed point theorem in [K] the claim follows. (It is also easy to prove it directly)

(3) $x = (x_1, x_2, \dots, x_n, \dots) \in \ell^2$.

$S_r x = (0, x_1, x_2, \dots)$

$S_\ell x = (x_2, x_3, \dots)$

Show

1) $\|S_r\| = \|S_\ell\| = 1$, none of S_r, S_ℓ compact

Sol: $\|S_r x\|_{\ell^2} = \|(0, x_1, x_2, \dots)\|_{\ell^2} = \|x\|_{\ell^2}$ so $\|S_r\| = 1$.

$\|S_\ell x\|_{\ell^2} = \|(x_2, x_3, \dots)\|_{\ell^2} \leq \|x\|_{\ell^2}$ so $\|S_\ell\| \leq 1$

But $\|S_\ell(0,1,0,\dots)\|_{\ell^2} = \|(1,0,0,\dots)\|_{\ell^2} = 1$ and $\|(0,1,0,\dots)\|_{\ell^2} = 1$ so $\|S_\ell\| = 1$.

S_r, S_ℓ cannot be compact because $S_\ell \circ S_r = I$ and if one or both of S_r, S_ℓ were compact then I would be compact, but I is not compact.

2) S_r has no eigenvalues

Sol: If $S_r x = \lambda x$ and $x = (x_1, x_2, \dots)$ then $(0, x_1, x_2, \dots) = \lambda(x_1, x_2, \dots)$ and so $x_1 = x_2 = \dots = x_n = 0$ (both for $\lambda \neq 0$ and $\lambda = 0$)

Hence S_r has no eigenvalue

3) $\sigma(S_r) = [-1, 1]$.

Sol: Clearly $\sigma(S_r) \subset [-1,1]$ since $\|S_r\| = 1$.

Consider $(S_r - \lambda I)(x_1, x_2, \dots, x_n, \dots) = (y_1, y_2, \dots, y_n, \dots)$

Then $x_n = -\frac{1}{\lambda} y_n - \frac{1}{\lambda^2} y_{n-1} - \dots - \frac{1}{\lambda^n} y_1$, $n = 1, 2, \dots$

for $\lambda \neq 0$. With $y_1 = -1$, $y_n = 0$, $n = 2, 3, \dots$

we see that $(S_r - \lambda I)$ is not surjective for $\lambda \in [-1,1]$ since $\sum_{n=1}^{\infty} (\frac{1}{|\lambda|^n})^2 = \infty$.

4/ $\lambda \in (-1,1)$ is an eigenvalue for S_ℓ.

Sol: $S_\ell x = \lambda x$ implies $x_n = \lambda^{n-1} x_1$, $n = 2, 3, \dots$

Hence $x = a(1, \lambda, \lambda^2, \dots, \lambda^{n-1}, \dots) \in \ell^2$ for all $\lambda \in (-1,1)$ and $a \in \mathbb{R}$. The eigenspace is given by $\{ a(1, \lambda, \lambda^2, \dots) : a \in \mathbb{R} \}$

5) $\sigma(S_\ell) = [-1,1]$

Sol: Note that $\sigma_p(S_\ell) = (-1,1) \subset \sigma(S_\ell) \subset [-\|S_\ell\|, \|S_\ell\|] = [-1,1]$ and $\sigma(S_\ell)$ closed. Hence $\sigma(S_\ell) = [-1,1]$.

4 See textbook

5 1/ $T_n \to T$ uniformly $\Rightarrow T_n \to T$ strongly

Sol: Fix $x \in X$. Then

$$\| T_n(x) - T(x) \|_Y = \| (T_n - T)(x) \|_Y \leq \| T_n - T \| \cdot \| x \|_X \to 0$$
$n \to \infty$

2/ $T_n \to T$ strongly $\not\Rightarrow T_n \to T$ uniformly

Sol: Assume $(x_n)_{n=1}^{\infty}$ is an ON-basis for a Hilbert space E. Set $T_n(x) = \langle x, x_n \rangle x_n$, $n = 1, 2, \dots$ Then

$T_n \to 0$ strongly since for every $x \in E$
$\| T_n x \|^2 = |\langle x, x_n \rangle|^2 \to 0$, $n \to \infty$
since $\sum_{n=1}^{\infty} |\langle x, x_n \rangle|^2 < \infty$

$T_n \not\to 0$ uniformly since $\| T_n \| = 1$ all n.

6 First part see textbook

Second part: $x_n \rightharpoonup x$ in H $\Rightarrow$ $\|x\| \leq \liminf_{n\to\infty} \| x_k \|$ ($k \geq n$)

Sol: For $y \in E$ we have

$|\langle x, y \rangle| = \lim_{n\to\infty} |\langle x_n, y \rangle|$ and so

$$|\langle y, x\rangle| = \lim_{n\to\infty} \inf_{k\geq n} |\langle y, x_k\rangle| \leq$$
$$\leq \left(\lim_{n\to\infty} \inf_{k\geq n} \|x_k\|\right) \|y\|$$

But $\langle \cdot, x\rangle : E \to \mathbb{C}$ is a bounded linear functional with operator norm $= \|x\|$.

Hence $\|x\| \leq \lim_{n\to\infty} \inf_{k\geq n} \|x_k\|$.