Solutions to TMA401/MMA400 , 2018-09-01

(1) $f \in C([0,1])$, $|\lambda| < \frac{1}{e-2}$. Show

$$\begin{cases} u''(x) + 2u'(x) + u(x) + \lambda \sin^2(u(x)) = f(x), & x \in [0,1] \\ u(0) = u(1) = 0 \end{cases}$$

has a unique solution $u \in C^2([0,1])$.

Solution:

Step 1: Calculate the Green's function:

Let $Lu = u'' + 2u' + u$. $N(L)$ has a basis

$u_1(x) = e^{-x}$, $u_2(x) = xe^{-x}$. Set $e(x,t) = a_1(t)u_1(x) + a_2(t)u_2(x)$

where

$$\begin{cases} 0 = e(t,t) = a_1(t)e^{-t} + a_2(t)te^{-t} \\ 1 = e'_x(t,t) = -a_1(t)e^{-t} + a_2(t)(e^{-t} - te^{-t}) \end{cases}$$

This implies $a_1(t) = -te^{t}$, $a_2(t) = e^{t}$. Now set

$g(x,t) = e(x,t)\Theta(x-t) + b_1(t)u_1(x) + b_2(t)u_2(x)$.

The conditions

$$\begin{cases} 0 = g(0,t) = b_1(t) \\ 0 = g(1,t) = -te^{t-1} + e^{t-1} + b_1(t)e^{-1} + b_2(t)e^{-1} \end{cases}$$

for $0<t<1$ gives $b_1(t) = 0$, $b_2(t) = (t-1)e^{t}$.

Hence the Green's function is given by

$$g(x,t) = \begin{cases} (t-1)xe^{t-x} & 0 \le x < t \le 1 \\ (x-1)te^{t-x} & 0 \le t < x \le 1 \end{cases}$$

Step 2: Prove that the BVP has a unique solution.

Set $T(u)(x) = \int_0^1 g(x,t)\,[f(t) - \lambda \sin^2(u(t))]\,dt$, $x \in [0,1]$,

for $u \in C([0,1])$. Let $(C([0,1]), \|\cdot\|)$ denote the normed space with $\|v\| = \max_{x\in[0,1]} |v(x)|$. This normed space is a Banach space. Moreover

$T: C([0,1]) \to C([0,1])$, actually $T(C([0,1])) \subset C^2([0,1])$

We try to apply Banach's fixed point theorem to prove that the BVP has a unique solution.

Fix $u, v \in C([0,1])$. For $x \in [0,1]$

$$|T(u)(x) - T(v)(x)| = \left|\int_0^1 g(x,t)\cdot\lambda\cdot(\sin^2(u(t)) - \sin^2(v(t)))\,dt\right| \le$$

$$\le |\lambda| \int_0^1 |g(x,t)|\,|\sin^2(u(t)) - \sin^2(v(t))|\,dt .$$

Here $|\sin^2(a) - \sin^2(b)| \le \max_{c\in\mathbb{R}} |\frac{d}{dx}\sin^2 x|\cdot|a-b| =$

$$= \max_{c\in\mathbb{R}} |\sin(2c)|\cdot|a-b| = |a-b| \quad \text{for } a,b \in \mathbb{R}.$$

We conclude that

$$|T(u)(x) - T(v)(x)| \le |\lambda| \int_0^1 |g(x,t)|\,dt\, \|u-v\|$$

Moreover $g(x,t) \le 0$ for $0 \le x,t \le 1$ and so

$\int_0^1 |g(x,t)|\,dt = \int_0^1 g(x,t)\cdot(-1)\,dt = j(x)$, where

$j'' + 2j' + j = -1$, $j(0) = j(1) = 0$. This implies

$j(x) = Ae^{-x} + Bxe^{-x} - 1$ with $A - 1 = 0 = Ae^{-1} + Be^{-1} - 1$

i.e. $A = 1$, $B = e-1$. Moreover $\underbrace{0 \le j(x) \le e-2}_{\text{crude estimate.}}$ for $0 \le x \le 1$.

We conclude that

$$\|T(u) - T(v)\| \le |\lambda|\cdot(e-2)\,\|u-v\| \quad \text{all } u,v \in C([0,1]).$$

Since $|\lambda| < \frac{1}{e-2}$ we conclude that T is a contraction on $(C([0,1]), \|\cdot\|)$. We have that the BVP has a unique solution $u \in C^2([0,1])$ by Banach's fixed point theorem.

(2) $Tf(x) = \int_{-\infty}^{\infty} \frac{1}{2} e^{-|x-t|} f(t)\,dt$, $x \in \mathbb{R}$, for $f \in L^2(\mathbb{R})$

Show that

a) $Tf \in L^2(\mathbb{R})$ for $f \in L^2(\mathbb{R})$

b) $T \in \mathcal{B}(L^2(\mathbb{R}))$ with $\|T\| \le 1$

c) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\frac{1}{2} e^{-|x-t|}|^2\,dx\,dt = \infty$

Solution: For $f \in L^2(\mathbb{R})$ we have

$$\int_{-\infty}^{\infty} |Tf(x)|^2\,dx \le \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \frac{1}{2} e^{-|x-t|}\,|f(t)|\,dt\right)^2 dx =$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \sqrt{\tfrac{1}{2} e^{-|x-t|}}\cdot\sqrt{\tfrac{1}{2} e^{-|x-t|}}\,|f(t)|\,dt\right)^2 dx \le \{\text{Hölder's} \le\}$$

$$\leq \int_{-\infty}^{\infty} \Big(\underbrace{\int_{-\infty}^{\infty} \tfrac{1}{2} e^{-|x-t|} dt}_{=1} \Big) \Big(\int_{-\infty}^{\infty} \tfrac{1}{2} e^{-|x-t|} |f(t)|^2 dt \Big) dx \leq$$

$$\leq \int_{-\infty}^{\infty} \Big(\int_{-\infty}^{\infty} \tfrac{1}{2} e^{-|x-t|} |f(t)|^2 dt \Big) dx =$$

$$= \int_{-\infty}^{\infty} \Big(\int_{-\infty}^{\infty} \tfrac{1}{2} e^{-|x-t|} |f(t)|^2 dx \Big) dt =$$

$$= \int_{-\infty}^{\infty} |f(t)|^2 \cdot \underbrace{\int_{-\infty}^{\infty} \tfrac{1}{2} e^{-|x-t|} dx}_{=1} \, dt = \int_{-\infty}^{\infty} |f(t)|^2 dt$$

Hence

(*) $\|Tf\|_{L^2(\mathbb{R})} \leq \|f\|_{L^2(\mathbb{R})}$, $f \in L^2(\mathbb{R})$. which proves a)

$T: L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is linear follows from

$$T(\alpha f + \beta g)(x) = \int_{-\infty}^{\infty} \tfrac{1}{2} e^{-|x-t|} (\alpha f(t) + \beta g(t)) dt =$$

$$= \alpha \int_{-\infty}^{\infty} \tfrac{1}{2} e^{-|x-t|} f(t) dt + \beta \int_{-\infty}^{\infty} \tfrac{1}{2} e^{-|x-t|} g(t) dt =$$

$$= \alpha Tf(x) + \beta Tg(x) = (\alpha Tf + \beta Tg)(x)$$

for all scalars α, β and $f, g \in L^2(\mathbb{R})$. Finally (*) above gives $\|T\| \leq 1$. which proves b)

Finally

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \big| \tfrac{1}{2} e^{-|x-t|} \big|^2 dx\, dt = \int_{-\infty}^{\infty} \Big(\tfrac{1}{4} \int_{-\infty}^{\infty} e^{-2|x-t|} dx \Big) dt =$$

$$= \tfrac{1}{4} \int_{-\infty}^{\infty} 1\, dt = \infty .$$ c) proven

(3) H Hilbert space, P, Q orthogonal projections on H.

Show: $P+Q$ orthogonal projection $\Rightarrow$ $P(H) \perp Q(H)$.

Solution: A bounded linear operator on a Hilbert space is an orthogonal projection iff it is self-adjoint and idempotent. So P, Q and $P+Q$ orthogonal projections implies P, Q self-adjoint, $P^2 = P$, $Q^2 = Q$ and $(P+Q)^2 = P+Q$

Hence $PQ + QP = 0$.

To show $Q(H) \perp P(H)$ fix $x \in Q(H)$ and show $x \in P(H)^\perp$

OPT implies $x = y + z$ where $y \in P(H)$ and $z \in P(H)^\perp$.

Apply P to x. This gives

$$Px = Py + Pz = Py = y$$

Apply PQ to x and use $PQ = -QP$. We obtain

$$Px = PQx = -QPx = -Qy$$

Hence $\quad y + Qy = 0$.

Apply Q implies $Qy = 0$ which gives $y = 0$.

Thus $\quad x = z \in P(H)^\perp \quad$ i.e $\quad Q(H) \subset P(H)^\perp$.

So $\quad Q(H) \perp P(H)$.

(4) & (5) : See textbook

(6) See [K] for a) and b).

c) With $X = \mathbb{R}$ and $\quad T(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$

we get $T^2(x) = 1$ for all $x \in \mathbb{R}$. So T^2 is a contraction with the unique fixed point 1.

Clearly $T : \mathbb{R} \to \mathbb{R}$ is not continuous ($\mathbb{R}$ equiped with the norm given by the abs. value.) but 1 is the unique fixed point for T.