TMA401 / MMA 2018-10-31

Solutions

(1) Show that $u''(x) + \sin^2(u(x)) = x$, $x \in [0,1]$ with $u(0) = u(1) = 0$ has a unique solution $u \in C^2([0,1])$.

Solution: We calculate the Green's function $g(x,t)$ for $L = (\frac{d}{dx})^2$ and $R_1 u = u(0) = 0$, $R_2 u = u(1) = 0$.

With $u_1(x) = 1$, $u_2(x) = x$ a basis for $N(L)$ we set

$$g(x,t) = \underbrace{(a_1(t)u_1(x) + a_2(t)u_2(x))}_{= e(x,t)}\Theta(x-t) + b_1(t)u_1(x) + b_2(t)u_2(x)$$

where

$$\begin{cases} e(t,t) = 0, \quad e'_x(t,t) = 1, & t \in [0,1] \\ R_1 g(\cdot,t) = R_2 g(\cdot,t) = 0, & t \in (0,1). \end{cases}$$

This yields $a_1(t) = -t$, $a_2(t) = 1$, $b_1(t) = 0$, $b_2(t) = t-1$ and we have

$$g(x,t) = (x-t)\Theta(x-t) + (t-1)x = \begin{cases} (t-1)x & 0 \le x < t \le 1 \\ (x-t)t & 0 \le t < x \le 1 \end{cases}$$

Now set $T(u)(x) = \int_0^1 g(x,t)(t - \sin^2(u(t)))\,dt$, $x \in [0,1]$.

Here $T: C([0,1]) \to C([0,1])$ and if $T$ has a unique fixed point in $C([0,1])$ then the BVP has a unique solution $u \in C^2([0,1])$. Assume that $C([0,1])$ is equipped with the max-norm, i.e. $\|h\| = \max_{0\le x\le 1} |h(x)|$, and so $(C([0,1]), \|\cdot\|)$ is a Banach space.

$T$ is a contraction on $C([0,1])$ since for $v, w \in C([0,1])$

$$|T(v)(x) - T(w)(x)| = \left|\int_0^1 g(x,t)\,(\sin^2(v(t)) - \sin^2(w(t)))\,dt\right| \le$$

$$\le \int_0^1 |g(x,t)|\,\underbrace{|\sin^2(v(t)) - \sin^2(w(t))|}_{\substack{\le \max_{c\in\mathbb{R}} |\sin(2c)|\,\|v-w\| = \|v-w\| \\ \text{by the mean value thm}}}\,dt \le$$

$$\le \int_0^1 |g(x,t)|\,dt \cdot \|v-w\|$$

We see that $g(x,t) \le 0$ and hence $j(x) \equiv \int_0^1 g(x,t)(-1)\,dt$ is a solution to $j''(x) = -1$, $j(0) = j(1) = 0$ i.e. $j(x) = \frac{1}{2}x(1-x)$

This gives $\max_{x\in[0,1]} j(x) = \frac{1}{8}$ and

$\| T(v) - T(w) \| \le \frac{1}{8} \| v - w \|$ for all $v, w \in C([0,1])$.

Banach's fixed point theorem now gives that $T$ has a unique fixed point and hence the BVP has a unique solution in $C^2([0,1])$.

② For $x = (x_1, x_2, \dots) \in l^2$ set $T(x) = y = (y_1, y_2, \dots)$ where $y_1 = x_1$, $y_n = \frac{1}{2^{n-1}}(x_1 + x_2 + \dots + x_n)$, $n = 2, 3, \dots$

Show that $T \in \mathcal{B}(l^2, l^2)$ and $T$ not surjective

Solution: For $x \in l^2$ we have

$$\| T(x) \|_{l^2}^2 = |x_1|^2 + \sum_{n=2}^{\infty} \Big( \frac{1}{2^{n-1}} \underbrace{|x_1 + x_2 + \dots + x_n|}_{\le \sum_{k=1}^{n} |x_k|} \Big)^2 \le$$

$$\le \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} \Big( \sum_{k=1}^{n} |x_k| \Big)^2 \le \Big\{ \Big( \sum_{k=1}^{n} |x_k| \Big)^2 \le n \sum_{k=1}^{n} |x_k|^2 \text{ by Hölder} \Big\} \le$$

$$\le \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{n}{2^{n-1}} |x_k|^2 = \sum_{k=1}^{\infty} \Big( \sum_{n=k}^{\infty} \frac{n}{2^{n-1}} \Big) |x_k|^2$$

Here $\sum_{n=k}^{\infty} \frac{n}{2^{n-1}} \le \sum_{n=1}^{\infty} \frac{n}{2^{n-1}}$ converges in $(\mathbb{R}, |\cdot|)$

e.g. by the root test $\sqrt[n]{\frac{n}{2^{n-1}}} \to \frac{1}{2}$, $n \to \infty$.

Set $M = \Big( \sum_{n=1}^{\infty} \frac{n}{2^{n-1}} \Big)^{1/2}$. Then

$\| T(x) \|_{l^2} \le M \| x \|_{l^2}$ for all $x \in l^2$.

To show that $T$ is not surjective we note that $y = (y_1, y_2, \dots)$ with $y_1 = 1$, $y_n = \frac{n}{2^{n-1}}$ $n = 2, 3, \dots$ we note that $y = T(x)$ where $x = (1, 1, \dots) \notin l^2$

③ $T: L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined by $T(f)(x) = f(x+1)$ all $x \in \mathbb{R}$. Show a) $T \in \mathcal{B}(L^2(\mathbb{R}), L^2(\mathbb{R}))$, calculate $\|T\|$

b) $T$ has no eigenvalues

Solution: a) $T$ linear: For $f, \tilde{f} \in L^2(\mathbb{R})$ and $\alpha, \tilde{\alpha}$ scalars

$$T(\alpha f + \tilde{\alpha} \tilde{f})(x) = (\alpha f + \tilde{\alpha} \tilde{f})(x+1) = \alpha f(x+1) + \tilde{\alpha} \tilde{f}(x+1) =$$

$$= \alpha T(f)(x) + \tilde{\alpha} T(\tilde{f})(x) = (\alpha T(f) + \tilde{\alpha} T(\tilde{f}))(x).$$

Hence $T$ linear.

$$\|T(f)\|_{L^2}^2 = \int_{\mathbb{R}} |T(f)(x)|^2 dx = \int_{\mathbb{R}} |f(x+1)|^2 dx = \int_{\mathbb{R}} |f(x)|^2 dx =$$
$$= \|f\|_{L^2}^2 .$$

Hence $T$ bounded and $\|T\| = 1$

b) Assume $\lambda$ eigenvalue with an eigenfunction $f$.

WLOG we assume $\|f\|_{L^2} = 1$.

Assume $|\lambda| \neq 1$: $f(x+1) = T(f)(x) = \lambda f(x)$ all $x \in \mathbb{R}$

implies $\|f\|_{L^2} = \|\lambda f\|_{L^2} = |\lambda| \|f\|_{L^2}$.

Hence, no eigenvalue with $|\lambda| \neq 1$

Assume $|\lambda| = 1$: Fix $0 < \varepsilon < \frac{1}{1+\sqrt{2}}$, see (*) below

$C_0(\mathbb{R})$ dense in $L^2(\mathbb{R})$. Choose $g \in C_0(\mathbb{R})$

s.th $\|f - g\|_{L^2} < \varepsilon$.

Since $g$ has compact support there exists $R > 0$

s.th $\operatorname{supp} g \equiv \{x \in \mathbb{R} : g(x) \neq 0\} \subset [-R, R]$.

Moreover note that for all positive integers $m$

$f(x+m) = \lambda^m f(x)$, all $x \in \mathbb{R}$.

Fix $m$ large enough s.th $\operatorname{supp} g \cap \operatorname{supp} g(\cdot + m) = \emptyset$

Then

$$\|f(\cdot+m) - \lambda^m f(\cdot)\|_{L^2} = \|f(\cdot+m) - g(\cdot+m) + g(\cdot+m) -$$
$$- \lambda^m g(\cdot) + \lambda^m g(\cdot) - \lambda^m f(\cdot)\|_{L^2} \geq \|g(\cdot+m) - \lambda^m g(\cdot)\|_{L^2} -$$
$$- (\|f(\cdot+m) - g(\cdot+m)\|_{L^2} + |\lambda|^m \|f(\cdot) - g(\cdot)\|_{L^2}) \geq$$
$$\geq \|g(\cdot+m) - \lambda^m g(\cdot)\|_{L^2} - 2\varepsilon =$$
$$= \left(\int_{\mathbb{R}} |g(x+m)|^2 dx + \int_{\mathbb{R}} (|\lambda^m| |g(x)|)^2 dx\right)^{1/2} - 2\varepsilon =$$
$$= \left(2\|g\|_{L^2}^2\right)^{1/2} - 2\varepsilon = \sqrt{2}\|g\|_{L^2} - 2\varepsilon = \sqrt{2}\left(\|g\|_{L^2} - \sqrt{2}\varepsilon\right)$$

But $\|g\|_{L^2} \geq \|f\|_{L^2} - \|f - g\|_{L^2} > 1 - \varepsilon$

Hence $\|f(\cdot+m) - \lambda^m f(\cdot)\|_{L^2} \geq \sqrt{2}(1 - \varepsilon - \sqrt{2}\varepsilon) =$

(*) $= \sqrt{2}(1 - (1+\sqrt{2})\varepsilon) > 0$ provided $0 < \varepsilon < \frac{1}{1+\sqrt{2}}$.

Contradiction. Hence $T$ has no eigenvalue with $|\lambda| = 1$.

(4) & (5) : See textbook and [K].

(6) $E$ complex Hilbert space and $T \in \mathcal{B}(E,E)$ with $\langle T(x), x\rangle \geq 0$ all $x \in E$.

Show $|\langle T(x), y\rangle|^2 \leq \langle T(x), x\rangle \cdot \langle T(y), y\rangle$ all $x, y \in E$.

Solution: From Homework assignment 3 we know that $T$ is self-adjoint. $\Big[\ \langle T(x), x\rangle = \langle x, T(x)\rangle =$

$= \langle T^*(x), x\rangle$ all $x \in E$ so $\langle (T-T^*)(x), x\rangle = 0$ all $x \in E$. Set $A = T - T^*$ and note that for all $x, y \in E$

$$\begin{cases} 0 = \langle A(x+y), x+y\rangle = \langle A(x), y\rangle + \langle A(y), x\rangle \\ 0 = \langle A(x+iy), x+iy\rangle = i\left(-\langle A(x), y\rangle + \langle A(y), x\rangle\right) \end{cases}$$

and so $\langle A(x), y\rangle = 0$ all $x, y \in E$. Hence $A(x) = 0$ all $x \in E$ and $A = 0$ $\Big]$

For $x, y \in E$ and $\alpha \in \mathbb{C}$ we obtain

$$0 \leq \langle T(\alpha x + y), \alpha x + y\rangle = \langle T(y), y\rangle + \alpha\langle T(x), y\rangle +$$

$$+ \bar{\alpha}\langle T(y), x\rangle + |\alpha|^2\langle T(x), x\rangle = \{T \text{ self-adjoint}\} =$$

$$= \langle T(y), y\rangle + \alpha\langle T(x), y\rangle + \bar{\alpha}\underbrace{\langle y, T(x)\rangle}_{\overline{\langle T(x), y\rangle}} + |\alpha|^2\langle T(x), x\rangle =$$

$$= \langle T(y), y\rangle + 2\operatorname{Re}\left(\alpha\langle T(x), y\rangle\right) + |\alpha|^2\langle T(x), x\rangle.$$

Assume $\langle T(x), y\rangle \neq 0$ otherwise there is nothing to prove.

Set $\alpha = e^{-i\arg\langle T(x), y\rangle} \cdot t$, $t \in \mathbb{R}$. Then

$$0 \leq \langle T(y), y\rangle + 2t|\langle T(x), y\rangle| + t^2\langle T(x), x\rangle \quad \text{all } t \in \mathbb{R}$$

Then $\langle T(x), x\rangle > 0$ since otherwise there will be a contradiction as $t \to -\infty$. Set $t = -\dfrac{|\langle T(x), y\rangle|}{\langle T(x), x\rangle}$ (minimizing the RHS). This gives

$$|\langle T(x), y\rangle|^2 \leq \langle T(x), x\rangle\langle T(y), y\rangle.$$

[K]