Written exam TMA 401/MMA 400, 2019-01-08

(1) Show that

$$(*) \quad \begin{cases} u''(x) + u(x) + \arctan(u(x)) = x, & x \in [0, \frac{\pi}{2}] \\ u(0) = 0, \; u(\frac{\pi}{2}) = \frac{\pi}{2} \end{cases}$$

has a unique solution $u \in C([0, \frac{\pi}{2}])$.

Solution: Set $u(x) = v(x) + x$, $x \in [0, \frac{\pi}{2}]$. Then

$$\begin{cases} v''(x) + v(x) = -\arctan(v(x) + x) \\ v(0) = v(\frac{\pi}{2}) = 0 \end{cases}$$

Set $Lv = v'' + v$, $R_1 v = v(0) = 0$, $R_2 v = v(\frac{\pi}{2}) = 0$

Step 1: Calculate the Green's function for L, R_1, R_2 above

$u_1(x) = \cos x$, $u_2(x) = \sin x$ is a basis for $N(L)$

$$g(x,t) = (a_1(t) u_1(x) + a_2(t) u_2(x))\,\Theta(x-t) + b_1(t) u_1(x) + b_2(t) u_2(x)$$

where

$$\begin{cases} a_1(t)\cos t + a_2(t)\sin t = 0 \\ -a_1(t)\sin t + a_2(t)\cos t = 1 \end{cases} \quad \text{i.e} \quad \begin{cases} a_1(t) = -\sin t \\ a_2(t) = \cos t \end{cases}$$

and

$$\begin{cases} b_1(t) = 0 \\ \cos t + b_2(t) = 0 \end{cases} \quad \text{i.e.} \quad \begin{cases} b_1(t) = 0 \\ b_2(t) = -\cos t \end{cases}$$

This yields

$$g(x,t) = \sin(x-t)\cdot\Theta(x-t) - \cos t \sin x =$$

$$= \begin{cases} -\sin t \cdot \cos x & 0 \le t < x \le 1 \\ -\cos t \cdot \sin x & 0 \le x < t \le 1 \end{cases}$$

Step 2: Set

$$T(v)(x) = \int_0^1 g(x,t)\left(-\arctan(v(t) + t)\right) dt, \quad x \in [0, \tfrac{\pi}{2}]$$

for $v \in C([0,\frac{\pi}{2}])$. Consider the Banach space $C([0,\frac{\pi}{2}])$ with the norm $\|h\| = \max_{x \in [0,\frac{\pi}{2}]} |h(x)|$.

If T is a contraction on the Banach space $(C([0,\frac{\pi}{2}]), \|\cdot\|)$ then T has a unique fixed point $\tilde{v}$ and so

$u(x) = \tilde{v}(x) + x$ is the unique solution for the BVP. (*)

Fix $v, w \in C([0,\frac{\pi}{2}])$,

$$|T(v)(x) - T(w)(x)| \le \int_0^1 |g(x,t)|\, |\arctan(w(t)+t) - \arctan(v(t)+t)|\,dt$$

$$\le \{\text{mean value theorem}\} \le \int_0^1 |g(x,t)|\,dt\ \|v-w\|$$

Hence $\|T(v) - T(w)\| \le \max_{x\in[0,\frac{\pi}{2}]} \int_0^1 |g(x,t)|\,dt \cdot \|v-w\|$.

Remains to show that $\max_{x\in[0,\frac{\pi}{2}]} \int_0^1 |g(x,t)|\,dt < 1$.

We observe that $g(x,t) \le 0$ for $0 \le x,t \le 1$. Set

$j(x) = \int_0^1 |g(x,t)|\,dt = \int_0^1 g(x,t)\cdot(-1)\,dt$, $x \in [0,\frac{\pi}{2}]$.

This implies that $j''(x) + j(x) = -1$, $j(0) = j(\frac{\pi}{2}) = 0$.

We obtain $j(x) = -1 + a\cos x + b\sin x$ where $a = b = 1$

Hence $j(x) = -1 + \cos x + \sin x = \sqrt{2}\sin(x + \frac{\pi}{4}) - 1$ which gives $0 \le j(x) \le \sqrt{2} - 1 < 1$. This concludes the solution.

(e) $T_c(f)(x) = |x-c|\cdot f(x)$, $x \in [0,1]$ for $f \in C([0,1])$.

$(C([0,1])$, max-norm$)$ Banach space

show $\mathcal{R}(T_c)$ closed $\iff$ $c \notin [0,1]$.

Proof:

Assume $c \notin [0,1]$. Then $\mathcal{R}(T_c) = C([0,1])$

since $h(x) = \frac{1}{|x-c|}\cdot f(x) \in C([0,1])$ for every $f \in C([0,1])$

and $T_c(h)(x) = |x-c|\cdot\frac{1}{|x-c|} f(x) = f(x)$.

Assume $c \in [0,1]$. Set $f(x) = \sqrt{|x-c|}$, $x \in [0,1]$.

Moreover set

$$h_n(x) = \begin{cases} \frac{1}{\sqrt{|x-c|}} & \text{for } |x-c| > \frac{1}{n},\ x \in [0,1] \\ \sqrt{n} & \text{for } |x-c| \le \frac{1}{n},\ x\in[0,1] \end{cases}$$

Then $h_n(x) \in C([0,1])$ for $n = 1, 2, \ldots$ and

$|T_c(h_n)(x) - f(x)| \le \frac{2}{\sqrt{n}} \to 0$, $n \to \infty$.

We have $f \in \overline{\mathcal{R}(T_c)}$. But $f \notin \mathcal{R}(T_c)$ since if $T_c(g) = f$ we have $|x-c|\cdot g(x) = \sqrt{|x-c|}$, $x \in [0,1]$

and hence $g(x) = \frac{1}{\sqrt{|x-c|}}$, $x \in [0,1]\setminus\{c\}$. We conclude that $g \notin C([0,1])$. Done

(3) $T \in \mathcal{B}(X)$, where X is a Banach space

Show a) $\sum_{n=0}^{\infty} \frac{T^n}{n!}$ converges in $\mathcal{B}(X)$

b) $\|T\| < \ln 2$ implies that $S_m = \sum_{n=0}^{m} \frac{T^n}{n!}$ is invertible in $\mathcal{B}(X)$ for $m = 1, 2, \ldots$

Proof: a) $\mathcal{B}(X)$ with the operator norm is a Banach space since X is a Banach space. Moreover $\sum_{n=0}^{\infty} \frac{t^n}{n!} = e^t \in \mathbb{R}$ for every $t \in \mathbb{R}$. To prove $\sum_{n=0}^{\infty} \frac{T^n}{n!}$ converges in $\mathcal{B}(X)$ it is enough to prove that $\sum_{n=0}^{\infty} \frac{T^n}{n!}$ is abs. conv. in $\mathcal{B}(X)$. Note that

$$\sum_{n=0}^{\infty} \left\|\frac{T^n}{n!}\right\| = \sum_{n=0}^{\infty} \frac{1}{n!}\|T^n\| \leq \sum_{n=0}^{\infty} \frac{\|T\|^n}{n!} = e^{\|T\|} < \infty$$

The statement in a) follows

b) Assume $\|T\| < \ln 2$. Then

$$\|I - S_m\| = \left\| -T - \frac{T^2}{2} - \ldots - \frac{T^m}{m!} \right\| \leq \|T\| + \ldots + \frac{\|T\|^m}{m!} \leq$$

$$\leq \sum_{n=1}^{\infty} \frac{\|T\|^n}{n!} = e^{\|T\|} - 1 < e^{\ln 2} - 1 = 1.$$

Hence $I - (I - S_m)$ is invertible in $\mathcal{B}(X)$, i.e. S_m is invertible in $\mathcal{B}(X)$.

(4) See textbook

(5) $k \in C([0,1] \times [0,1])$. Set $T(f)(x) = \int_0^1 k(x,y) f(y)\, dy$, $x \in [0,1]$

Show a) $T \in \mathcal{B}(C([0,1]))$, $C([0,1])$ equipped with max-norm

b) $\|T\| = \max_{x \in [0,1]} \int_0^1 |k(x,y)|\, dy$

Proof: a)

- $f \in C([0,1]) \Rightarrow T(f) \in C([0,1])$: follows easily since k unif. cont. on the compact set $[0,1] \times [0,1]$
- T linear i.e. $T(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 T(f_1) + \alpha_2 T(f_2)$

- T bounded, i.e. $\exists M > 0$: $\|T(f)\| \le M\|f\|$ all $f \in C([0,1])$

follows easily since k is bounded on the compact set $[0,1]\times[0,1]$ since k continuous

b) We note that

$$|T(f)(x)| \le \int_0^1 |k(x,y)|\cdot|f(y)|\,dy \le \int_0^1 |k(x,y)|\,dy\,\|f\|\ , x\in[0,1]$$

This implies $\|T(f)\| \le \max_{x\in[0,1]} \int_0^1 |k(x,y)|\,dy \cdot \|f\|$ and

hence $\|T\| \le \max_{x\in[0,1]} \int_0^1 |k(x,y)|\,dy$.

To prove equality set

$$f_n^{(x)}(y) = \frac{|k(x,y)|}{|k(x,y)| + \frac{1}{n}}\ , \quad y\in[0,1],\ n=1,2,\ldots \quad \text{for fixed } x\in[0,1]$$

Here $f_n^{(x)} \in C([0,1])$ for $n=1,2,\ldots$ for every $x\in[0,1]$.

and $|f_n^{(x)}(y)| \le 1$ all $y\in[0,1]$.

We obtain

$$\|T\| \ge |T(f_n^{(x)})(x)| = \int_0^1 \frac{|k(x,y)|^2}{|k(x,y)|+\frac{1}{n}}\,dy \to \int_0^1 |k(x,y)|\,dy \quad n\to\infty$$

by Lebesgue dominated convergence thm (or in the Riemann setting by Dini theorem ($[0,1]$ compact set and $C([0,1]) \ni \frac{|k(x,y)|^2}{|k(x,y)|+\frac{1}{n}} \uparrow |k(x,y)| \in C([0,1])$ and hence $f_n^{(x)} \to |k(x,\cdot)|$ uniformly on $[0,1]$ for every $x\in[0,1]$, i.e. $\|f_n^{(x)} - |k(x,\cdot)|\| \to 0$, $n\to\infty$) since then interchanging of order of taking limit and integration is valid)

Hence $\|T\| \ge \int_0^1 |k(x,y)|\,dy$ for every $x\in[0,1]$

and so $\|T\| \ge \sup_{x\in[0,1]} \int_0^1 |k(x,y)|\,dy = \max_{x\in[0,1]} \int_0^1 |k(x,y)|\,dy$

Conclusion: $\|T\| = \max_{x\in[0,1]} \int_0^1 |k(x,y)|\,dy$.

⑥ $(X, \|\cdot\|)$ normed space. Show that a) $\Leftrightarrow$ b), where

a) $\|\cdot\|$ is induced by an inner product on X

b) $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$ all $x,y\in X$

Proof:

a) $\Rightarrow$ b): If $\|\cdot\|$ is induced by $\langle\cdot,\cdot\rangle$ then $\|\cdot\|$ satisfies the $\|$-law. Easy calculation $\|x+y\|^2 + \|x-y\|^2 =$

$= \langle x+y, x+y\rangle + \langle x-y, x-y\rangle$ and expand ...

b) $\Rightarrow$ a): Note that if $\|\cdot\|$ is induced by $\langle\cdot,\cdot\rangle$ then

$\langle x,y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2)$ (well-known fact)

Consider $F(x,y) = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2)$, $x,y \in X$

and show that $F: X \times X \to \mathbb{R}$ satisfies the

inner product axioms, i.e.

1) $F(x,x) \geq 0$ all $x \in X$, $= 0 \Rightarrow x = 0 \in X$

2) $F(\alpha x + \beta y, z) = \alpha F(x,z) + \beta F(y,z)$ all $x,y,z \in X$, $\alpha,\beta \in \mathbb{R}$

3) $F(x,y) = F(y,x)$ all $x,y \in X$.

Here 1) and 3) follows easily

(*) To prove prove that $F(x+y, z) = F(x,z) + F(y,z)$ all $x,y,z \in X$. If this holds then

$F(nx, z) = n F(x,z)$ all $x,z \in X$, n pos. integer

and also $F(x,z) = F(n \cdot \frac{1}{n} x, z) = n \cdot F(\frac{1}{n}x, z)$ i.e

$F(\frac{1}{n}x, z) = \frac{1}{n} F(x,z)$ all $x,z \in X$, n pos. integer

Also $F(0,z) = 0 F(x,z)$ all $x,z \in X$.

We have $F(rx, z) = r F(x,z)$ all $x,z \in X$, $r \in \mathbb{Q}$.

Also $F(rx,z) = r F(x,z)$ all $x,z \in X$, $r \in \mathbb{R}$ since

$$F(rx,z) = \frac{1}{4}(\|rx+z\|^2 - \|rx-z\|^2) =$$

$$= \frac{1}{4}\Big(\lim_{\substack{q\to r\\ q\in\mathbb{Q}}} \|qx+z\|^2 - \lim_{\substack{q\to r\\ q\in\mathbb{Q}}} \|qx-z\|^2\Big) =$$

$$= \lim_{\substack{q\to r\\ q\in\mathbb{Q}}} \Big(\frac{1}{4}(\|qx+z\|^2 - \|qx-z\|^2)\Big) =$$

$$= \lim_{\substack{q\to r\\ q\in\mathbb{Q}}} F(qx,z) = \lim_{\substack{q\to r\\ q\in\mathbb{Q}}} q F(x,z) = r F(x,z).$$

Remains to prove (*):

$$4(F(x+y,z) - F(x,z) - F(y,z)) =$$

$$= \|x+y+z\|^2 - \|x+y-z\|^2 - \|x+z\|^2 + \|x-z\|^2 - \|y+z\|^2 + \|y-z\|^2 =$$

$$= \{\text{//-law}\} = \|x+y+z\|^2 - \big(-\|x-y-z\|^2 + 2(\|x-z\|^2 + \|y\|^2)\big) -$$

$$- \|x+z\|^2 + \|x-z\|^2 - \|y+z\|^2 + \|y-z\|^2 = \{\text{//-law}\} =$$

$$= 2(\|x\|^2 + \|z+y\|^2) - 2(\|x-z\|^2 + \|y\|^2) - \|x+z\|^2 + \|x-z\|^2 -$$

$$- \|y+z\|^2 + \|y-z\|^2 = 2(\|x\|^2 - \|y\|^2) + \|z+y\|^2 + \|z-y\|^2 -$$

$$- \|x+z\|^2 - \|x-z\|^2 = \{\text{$\|$-law}\} = 2(\|x\|^2 - \|y\|^2) + 2(\|z\|^2 + \|y\|^2) -$$

$$-2(\|x\|^2 + \|z\|^2) = 0.$$

Done.