MATEMATIK
Chalmers tekniska högskola och Göteborgs universitet

Functional Analysis ENM, TMA401/ Applied Functional Analysis GU, MMA400,
Date: 2012-10-24 (4 hours)

Aids: Just pen, ruler and eraser.
Teacher on duty: Christoffer Standar, 0703-088304

Note:	Write your name and personal number on the cover. Write your code on every sheet you hand in. Only write on one page of each sheet. Do not use red pen. Do not answer more than one question per page. State your methodology carefully. Write legibly. Questions are not numbered by difficulty. Sort your solutions by the order of the questions. Mark on the cover the questions you have answered. Count the number of sheets you hand in and fill in the number on the cover. To pass requires 10 points.

1. Consider the following boundary value problem

$$\begin{cases} (Lu)(x) \equiv u''(x) = f(x, u(x)),\ 0 \leq x \leq 1, \\ u(0) = u(1) = 1, \\ u \in C^2([0,1]). \end{cases}$$

   (a) Calculate the Green's function

   (b) Calculate the eigenfunctions for the differential operator L with the boundary conditions above. What could be said about the sequence of all eigenfunctions?

   (c) Show that the boundary value problem has a unique solution for $f(x,u) = 15x\frac{u}{1+u^2}$.

(5p)

2. Let E be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and corresponding norm $\| \cdot \|$. Assume that N is a positive integer and $\alpha \in \mathbb{C}$ with $\alpha^2 \neq 1$ and $\alpha^N = 1$. Show that

$$\langle x, y \rangle = \frac{1}{N} \Sigma_{n=1}^{N} \|x + \alpha^n y\|^2 \alpha^n.$$

(3p)

3. Let $1 < p < \infty$ and $q = \frac{p}{p-1}$. Show that for every $\mathbb{y} = (y_1, y_2, y_3, \ldots) \in l^q$ the mapping $f_{\mathbb{y}}$ given by $f_{\mathbb{y}}(\mathbb{x}) = \Sigma_{n=1}^{\infty} x_n y_n$, $\mathbb{x} = (x_1, x_2, x_3, \ldots)$, defines a bounded linear mapping $l^p \to \mathbb{C}$. Calculate the operator norm for $f_{\mathbb{y}}$. Finally show that for every bounded linear mapping $f : l^p \to \mathbb{C}$ there exists a $\mathbb{y} \in l^q$ such that $f = f_{\mathbb{y}}$.

(4p)

4. Formulate the following theorems:

   (a) Banach-Steinhaus theorem

   (b) Schauder's fixed point theorem

   (c) Lax-Milgram theorem

   (d) Hilbert-Schmidt theorem

(5p)

5. Let X be a Banach space with norm $\|\cdot\|$. Let T be a bounded linear mapping $X \to X$. Define the operator norm $\|T\|_{X \to X}$ and show that this defines a norm on the vector space $\mathcal{B}(X, X)$ of all bounded linear mappings $X \to X$. Finally show that $\mathcal{B}(X, X)$ with the norm $\|\cdot\|_{X \to X}$ is a Banach space.

(4p)

6. Let A be a compact linear mapping $E \to E$ in a separable Hilbert space E. Show that there is a sequence of finite-rank operators A_n, $n = 1, 2, 3, \ldots$, on E such that

$$\|A - A_n\|_{E \to E} \to 0, \quad \text{as } n \to \infty.$$

(4p)

For information on the announcement of results see the course homepage where also solutions to the problems will be presented.

GOOD LUCK! PK