MATEMATIK
Chalmers tekniska högskola och Göteborgs universitet

Functional Analysis ENM, TMA401/ Applied Functional Analysis GU, MMA400,
Date: 2013-08-31 (4 hours)

Aids: Just pen, ruler and eraser.
Teacher on duty: Anders Martinsson, 0703-088304

Note:	Write your name and personal number on the cover. Write your code on every sheet you hand in. Only write on one page of each sheet. Do not use red pen. Do not answer more than one question per page. State your methodology carefully. Write legibly. Questions are not numbered by difficulty. Sort your solutions by the order of the questions. Mark on the cover the questions you have answered. Count the number of sheets you hand in and fill in the number on the cover. To pass requires 10 points.

1. Show that the boundary value problem

$$\begin{cases} u''(x) + 2 + \frac{1}{1+u^2(x)} = 0, \quad x \in [0,1], \\ u(0) = u(1) = 0, \\ u \in C^2([0,1]). \end{cases}$$

has a unique solution u.

(4p)

2. Set

$$Tf(x) = \int_0^{1-x} f(y)\,dy, \quad f \in C([0,1])$$

for $x \in [0,1]$. Prove that

(a) T defines a linear bounded and compact[1] operator on $C([0,1])$ (with the uniform norm), and

(b) calculate[2] $\sigma(T)$ and the eigenvalues of T.

(5p)

3. Let $(x_n)_{n=1}^{\infty}$ be an ON-sequence in a Hilbert space H and let $(c_n)_{n=1}^{\infty}$ be a sequence of complex numbers. Define T by

$$T(x) = \Sigma_{n=1}^{\infty} c_n \langle x, x_n \rangle x_n, \ x \in H.$$

[1] The definition of a compact operator on a Banach space is word by word the same as the definition of a compact operator on a Hilbert space.

[2] You may assume that $\sigma(T) \setminus \{0\} \subset \sigma_p(T)$ for a compact operator on a Banach space which indeed is true.

Give necessary and sufficient conditions on $(c_n)_{n=1}^{\infty}$ for T to be a well-defined bounded linear operator on H. Moreover give necessary and sufficient conditions on $(c_n)_{n=1}^{\infty}$ for T to be a compact operator on H. Prove your statements.

(4p)

4. State and prove the Hilbert-Schmidt theorem. Propositions that are used in the proof should be properly stated but need not be proven.

(5p)

5. Let H be a Hilbert space and $A \in \mathcal{B}(H, H)$. Define the adjoint operator A^*, show that it is a uniquely defined mapping in $B(H, H)$ and that $\|A^*\| = \|A\|$.

(4p)

6. Let A be a self-adjoint operator on a Hilbert space H and let $\lambda \in$. Show that $\lambda \notin \sigma(A)$ if and only if there exists a $C > 0$ such that

$$\|x\| \leq C\|Ax - \lambda x\| \ \text{ all } x \in H.$$

(4p)

For information on the announcement of results see the course homepage where also solutions to the problems will be presented.

GOOD LUCK! PK