MATEMATIK
Chalmers tekniska högskola och Göteborgs universitet

Functional Analysis ENM, TMA401/ Applied Functional Analysis GU, MMA400,
Date: 2013-10–23 (4 hours)

Aids: Just pen, ruler and eraser.
Teacher on duty: Magnus Önnheim, 0703-088304

Note:	Write your name and personal number on the cover. Write your code on every sheet you hand in. Only write on one page of each sheet. Do not use red pen. Do not answer more than one question per page. State your methodology carefully. Write legibly. Questions are not numbered by difficulty. Sort your solutions by the order of the questions. Mark on the cover the questions you have answered. Count the number of sheets you hand in and fill in the number on the cover. To pass requires 10 points.

1. Show that the boundary value problem

$$\begin{cases} u''(x) + u(x) + \lambda \cos(1 + u(x)) = 0, \quad x \in [0,1], \\ u(0) = u(1) = 0, \\ u \in C^2([0,1]). \end{cases}$$

has a unique solution u for $|\lambda| < \epsilon$, ϵ small. Give an upper bound on ϵ. Show also that the there exists a solution for arbitrary λ.

(4p)

2. Consider the vector space l^1 and set

$$\|\mathbb{x}\|_* = 2|\Sigma_{n=1}^{\infty} x_n| + \Sigma_{n=2}^{\infty}(1 + \frac{1}{n})|x_n|$$

for $\mathbb{x} = (x_1, x_2, \ldots, x_n, \ldots) \in l^1$. Show that $\|\mathbb{x}\|_*$ defines a norm on l^1 and that the vector space l^1 is a Banach space with this norm. Is this norm equivalent to the standard norm $\|\mathbb{x}\|_{l^1}$?

(5p)

3. Let $(e_n)_{n=1}^{\infty}$ be an ON-basis in a Hilbert space H and set $f_n = e_{n+1} - e_n$ for $n = 1, 2, 3, \ldots$ Show that $\mathrm{Span}\{f_n : n = 1, 2, 3, \ldots\}$ is dense in H.

(4p)

4. State Banach´s fixed point theorem and Schauder´s fixed point theorem. Prove Banach´s fixed point theorem.

(5p)

P.T.O.

5. Let H be a Hilbert space and $A \in \mathcal{B}(H, H)$. Show that the adjoint operator A^* is compact if A is compact.

(4p)

6. Let A be a self-adjoint operator on a Hilbert space H and let $\lambda \in \mathbb{C}$. Show that $\lambda \notin \sigma(A)$ if and only if there exists a $C > 0$ such that
$$\|x\| \leq C\|Ax - \lambda x\| \ \text{ all } x \in H.$$

(4p)

For information on the announcement of results see the course homepage where also solutions to the problems will be presented.

GOOD LUCK! PK