MATEMATIK
Chalmers tekniska högskola och Göteborgs universitet

Functional Analysis ENM, TMA401/ Applied Functional Analysis GU, MMA400,
Date: 2016-01–07 (4 hours)

Aids: Just pen, ruler and eraser.
Teacher on duty: Gustav Kettil, 0703-088304

Note:	Write your name and personal number on the cover. Write your code on every sheet you hand in. Only write on one page of each sheet. Do not use red pen. Do not answer more than one question per page. State your methodology carefully. Write legibly. Questions are not numbered by difficulty. Sort your solutions by the order of the questions. Mark on the cover the questions you have answered. Count the number of sheets you hand in and fill in the number on the cover. To pass requires 10 points.

1. Show that the BVP

$$\begin{cases} u''(x) + 2u'(x) + u(x) = \dfrac{1}{2}\sin^2(u(x)), \quad 0 \le x \le 1 \\ u(0) = u(1) = 0 \end{cases}$$

has a unique solution $u \in C^2([0,1])$.

(4p)

2. Set

$$Af(x) = \int_0^1 \sinh(x-t) f(t)\, dt, \quad 0 \le x \le 1.$$

Show that A is a linear bounded and compact operator on the Banach spaces

(a) $C([0,1])$
(b) $L^2([0,1])$

respectively (with the standard norms). Also calculate the operator norms.

(4p)

3. Set $\mathbb{R}_+ = \{x \in \mathbb{R} : x \ge 0\}$. For $f \in L^2(\mathbb{R}_+)$ define

$$Bf(x) = \frac{1}{x}\int_0^x f(t)\, dt, \quad x > 0.$$

Show that B is a bounded linear operator on $L^2(\mathbb{R}_+)$. Calculate the adjoint operator.

(4p)

4. State and prove the Hilbert-Schmidt theorem. Propositions and Lemmas used inte proof must be properly stated but need not be proven.

   (5p)

5. Let X and Y be normed spaces with $\dim X < \infty$. Show that every linear mapping $C : X \to Y$ is continuous.

   (4p)

6. Show that the interior of any compact set in an ∞–dimensional normed space is empty.

   (4p)

For information on the announcement of results see the course homepage where also solutions to the problems will be presented.

GOOD LUCK! PK