MATEMATIK
Chalmers tekniska högskola och Göteborgs universitet

Functional Analysis ENM, TMA401/ Applied Functional Analysis GU, MMA400,
Date: 2016–08–27 (4 hours)

Aids: Just pen, ruler and eraser.
Teacher on duty: Mathias Lennartsson, 5325

Note:	Write your name and personal number on the cover. Write your code on every sheet you hand in. Only write on one page of each sheet. Do not use red pen. Do not answer more than one question per page. State your methodology carefully. Write legibly. Questions are not numbered by difficulty. Sort your solutions by the order of the questions. Mark on the cover the questions you have answered. Count the number of sheets you hand in and fill in the number on the cover. To pass requires 10 points.

1. Show that the BVP

$$\begin{cases} u''(x) + \sin^2(u(x)) = 1, & 0 \le x \le 1 \\ u(0) = u(1) = 0 \end{cases}$$

has a unique solution $u \in C^2([0,1])$.

(4p)

2. Let $(x_n)_{n=1}^{\infty}$ be a converging sequence in a normed space. Call the limit x. Set

$$y_n = \frac{x_1 + 2x_2 + 3x_3 + \ldots + nx_n}{n^2}, \quad n = 1, 2, 3, \ldots$$

Prove that $(y_n)_{n=1}^{\infty}$ converges and find the limit.

(4p)

3. Let H be a Hilbert space with an orthonormal basis $(e_n)_{n=1}^{\infty}$. Let $T \in \mathcal{B}(H)$ be given by

$$T(\Sigma_{n=1}^{\infty} a_n e_n) = \Sigma_{n=1}^{\infty} \lambda_n a_n e_n.$$

Show that T is unitary[1] if and only if $|\lambda_n| = 1$ for all n.

(4p)

[1] T is called unitary if $T^*T = TT^* = I$

4. State and prove the Hilbert-Schmidt theorem. Propositions and Lemmas used in the proof must be properly stated but need not be proven.

(5p)

5. Let X be a Banach space, Y normed space and $T : X \to Y$ a bounded linear mapping. Assume that there exists a $C > 0$ such that
$$\|x\| \le C\|T(x)\| \text{ for all } x \in X.$$
Show that $\operatorname{Im} T$, that is the range of T, is also a Banach space, and hence in particular a closed subspace of Y.

(4p)

6. Let T be a compact self-adjoint operator on a Hilbert space H. Show[2] that
$$\sup\{|\lambda| : \lambda \in \sigma(T)\} = \|T\|.$$

(4p)

For information on the announcement of results see the course homepage where also solutions to the problems will be presented.

GOOD LUCK! PK

[2]The fact that there exists an eigenvalue λ to T with absolute value equal to the operator-norm of T must be proven.