MATEMATIK
Chalmers tekniska högskola och Göteborgs universitet

Functional Analysis ENM, TMA401/ Applied Functional Analysis GU, MMA400,
Date: 2016–10–26 (4 hours)

Aids: Just pen, ruler and eraser.
Teacher on duty: Jakob Björnberg

Note:	Write your name and personal number on the cover. Write your code on every sheet you hand in. Only write on one page of each sheet. Do not use red pen. Do not answer more than one question per page. State your methodology carefully. Write legibly. Questions are not numbered by difficulty. Sort your solutions by the order of the questions. Mark on the cover the questions you have answered. Count the number of sheets you hand in and fill in the number on the cover. To pass requires 10 points.

1. Show that the BVP

$$\begin{cases} (e^x u'(x))' + \dfrac{|u(x)|}{3+(u(x))^2} = 0, \ x \in [0,1] \\ \\ u(0) + u'(0) = 0, \ u(1) - u'(1) = 0 \end{cases}$$

has a unique solution $u \in C^2([0,1])$.

(4p)

2. Let $E = \{\mathbb{x} = (x_1, x_2, x_3, \ldots) \in l^2 : x_k \neq 0 \ \text{ only for finitely many } k\}$ equipped with the usual l^2 inner product. Set $M = \{\mathbb{x} \in E : \Sigma_{n=1}^{\infty} x_n = 0\}$. Determine $\overline{M}$ and $M^{\perp}$.

(4p)

3. Let $\mathbb{x} = (x_1, x_2, x_3, \ldots)$ be a sequence of non-negative real number and assume that $1 < p < \infty$. Moreover assume that

$$\Sigma_{n=1}^{\infty} |x_n y_n| < \infty \ \text{ for all } \mathbb{y} = (y_1, y_2, y_3, \ldots) \in l^q$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Show[1] that $\mathbb{x} \in l^p$.

(4p)

[1]Hint: Might be helpful to consider mappings

$$l^q \ni \mathbb{y} \mapsto \Sigma_{n=1}^{N} x_n y_n \in \mathbb{R}, \quad N = 1, 2, 3, \ldots.$$

4. Let A be a self-adjoint operator on a Hilbert space E. Prove that

$$\|A\| = \sup_{0 \neq x \in E} \frac{|\langle A(x), x\rangle|}{\|x\|^2}.$$

(5p)

5. (a) Let X be a Banach space and $T : X \to X$ a bounded linear mapping. Assume that there exists a $C > 0$ such that

$$\|x\| \leq C\|T(x)\| \text{ for all } x \in X.$$

Show that $\mathcal{R}(T)$ a closed subspace of X.

(b) Set $X = l^\infty$ with the the standard norm

$$\|(x_1, x_2, \ldots, x_n, \ldots)\|_{l^\infty} = \sup_{n=1,2,\ldots} |x_n|.$$

Let

$$T((x_1, x_2, \ldots, x_n, \ldots)) = (\frac{x_1}{1}, \frac{x_2}{2}, \ldots, \frac{x_n}{n}, \ldots).$$

Show that T is a bounded linear operator on l^∞ and that

$$T(X) = \{\mathbb{x} \in l^\infty : \sup_{n=1,2,\ldots} |n\, x_n| < \infty\}.$$

Set

$$\mathbb{y}_N = (\frac{y_1}{1}, \frac{y_2}{\sqrt{2}}, \ldots, \frac{y_N}{\sqrt{N}}, 0, 0, \ldots) \text{ for } N = 1, 2, 3, \ldots$$

Show that there exists an $\mathbb{y} = (y_1, y_2, y_3, \ldots) \in l^\infty$ such that $\mathbb{y} \notin T(X)$ but

$$T(X) \ni \mathbb{y}_N \to \mathbb{y} \text{ in } l^\infty$$

i.e. $\mathcal{R}(T)$ is not a closed subspace of l^∞.

(3+1p)

6. Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be Banach spaces. Assume that $K \in \mathcal{K}(X, Y)$. Moreover assume that $A \in \mathcal{B}(Y, Z)$ and that A is injective. Show that for every $\epsilon > 0$ there exists a real number $C_\epsilon > 0$ such that for all $x \in X$

$$\|K(x)\|_Y \leq \epsilon\|x\|_X + C_\epsilon\|AK(x)\|_Z.$$

(4p)

For information on the announcement of results see the course homepage where also solutions to the problems will be presented.

GOOD LUCK! PK