MATEMATIK
Chalmers tekniska högskola och Göteborgs universitet

Functional Analysis ENM, TMA401/ Applied Functional Analysis GU, MMA400,
Date: 2017–08–26 (4 hours)

Aids: Just pen, ruler and eraser.
Teacher on duty: Raad Salman, 5325

Note:	Write your name and personal number on the cover. Write your code on every sheet you hand in. Only write on one page of each sheet. Do not use red pen. Do not answer more than one question per page. State your methodology carefully. Write legibly. Questions are not numbered by difficulty. Sort your solutions by the order of the questions. Mark on the cover the questions you have answered. Count the number of sheets you hand in and fill in the number on the cover. To pass requires 10 points.

1. Show that the BVP

$$\begin{cases} 5u'' + \dfrac{1}{1+u^4(x)} = 1, \ x \in [0,1] \\ \\ u(0) = 0, \, u(1) = 0 \end{cases}$$

has a unique solution $u \in C^2([0,1])$.

(4p)

2. Set

$$Tf(x) = \int_{-\infty}^{\infty} \frac{f(y)}{1+(x-y)^2}\, dy, \ f \in L^2(\mathbb{R}).$$

Prove that T defines a bounded linear self-adjoint operator on $L^2(\mathbb{R})$. Show that T is not a compact operator.

(4p)

3. Let $x_n \in \mathbb{R}$ for $n = 1, 2, 3, \ldots$. Assume that $(x_n)_{n=1}^{\infty}$ has no cluster point and that $x_n \neq x_m$ for $n \neq m$. Moreover assume that and $(c_n)_{n=1}^{\infty} \in l^1$. Let $C(\mathbb{R})$ denote the vector space of all bounded continuous functions on $\mathbb{R}$ with norm $\|f\| = \sup_{x\in\mathbb{R}} |f(x)|$. Finally let L denote the functional on $C(\mathbb{R})$ defined by

$$L(f) = \Sigma_{n=1}^{\infty} c_n f(x_n), \ f \in C(\mathbb{R}).$$

Calculate $\|L\|$.

(4p)

4. State and prove the Orthogonal Projection Theorem. Also the the "Closest Point Property" theorem should be proved if used.

(5p)

5. Let $T \neq 0$ be a linear functional on a normed space X. Prove that the following statements are equivalent:

   (a) T is continuous

   (b) $\mathcal{N}(T)$ is a proper closed subspace of X

   (c) $\mathcal{N}(T)$ is not dense in X

(4p)

6. Let M be a dense subspace of a separable Hilbert space H. Prove that H has an ON-basis consisting of elements in M.

(4p)

For information on the announcement of results see the course homepage where also solutions to the problems will be presented.

GOOD LUCK! PK