MATEMATIK
Chalmers tekniska högskola och Göteborgs universitet

Functional Analysis ENM, TMA401/ Applied Functional Analysis GU, MMA400,
Date: 2017–10–25 (4 hours)

Aids: Just pen, ruler and eraser.
Teacher on duty: Christoffer Stander, 5325

Note:	Write your name and personal number on the cover.
	Write your code on every sheet you hand in.
	Only write on one page of each sheet. Do not use red pen.
	Do not answer more than one question per page.
	State your methodology carefully. Write legibly.
	Questions are not numbered by difficulty.
	Sort your solutions by the order of the questions.
	Mark on the cover the questions you have answered.
	Count the number of sheets you hand in and fill in the number on the cover.
	To pass requires 10 points.

1. Let $f \in C([0,1])$ and $\lambda \in \mathbb{R}$ with $|\lambda| < e(e-1)$. Show that the boundary value problem

$$\begin{cases} u'' + u' + \lambda |u(x)| = f(x), \ x \in [0,1] \\ \\ u(0) = u(1) = 0 \end{cases}$$

has a unique solution $u \in C^2([0,1])$.

(4p)

2. Show that the mapping $A : C([0,1]) \to C([0,1])$ given by

$$A(f)(x) = \int_0^x f(y)\,dy - \int_x^1 f(y)\,dy, \ x \in [0,1]$$

is a bounded linear mapping and calculate the operator norm $\|A\|$.

(4p)

3. Show[1] that the integral operator $A : C([0,1]) \to C([0,1])$ defined by

$$A(f)(x) = \int_0^1 k(x,y) f(y)\,dy, \ x \in [0,1],$$

where $k(x,y) \in C([0,1] \times [0,1])$ is strictly positive, has a positive eigenvalue with an strictly positive eigenfunction.

(4p)

[1] Hint: Use Schauder's fixed point theorem

4. Let E be a separable infinite-dimensional Hilbert space and let A be a compact self-adjoint operator on E. Prove that there exists a complete ON-sequence of eigenvectors of A in E, i.e. prove the spectral theorem for compact self-adjoint operators. The Hilbert-Schmidt theorem used in the proof should be stated and proved but other proposition used in the proof need not be proven.

(5p)

5. (a) Let E be a Hilbert space. Give the definition of a compact operator A on E.

   (b) Let A be a bounded linear operator a Hilbert space E. Show that A is a compact operator if A^*A is a compact operator on E.

(1+3p)

6. Let $(x_n)_{n=1}^{\infty}$ be a bounded sequence in a Hilbert space E. Show that there exists a subsequence $(x_{n_k})_{k=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ that converges weakly in E.

(4p)

For information on the announcement of results see the course homepage where also solutions to the problems will be presented.

GOOD LUCK! PK