MATEMATIK
Chalmers tekniska högskola och Göteborgs universitet

Functional Analysis ENM, TMA401/ Applied Functional Analysis GU, MMA400,
Date: 2018–09–01 (4 hours)

Aids: Just pen, ruler and eraser.
Teacher on duty: Andreas Petersson, 5325

Note:	Write your name and personal number on the cover. Write your code on every sheet you hand in. Only write on one page of each sheet. Do not use red pen. Do not answer more than one question per page. State your methodology carefully. Write legibly. Questions are not numbered by difficulty. Sort your solutions by the order of the questions. Mark on the cover the questions you have answered. Count the number of sheets you hand in and fill in the number on the cover. To pass requires 10 points.

1. Let $f \in C([0,1])$ and $\lambda \in \mathbb{R}$ with $|\lambda| < \frac{1}{e-2}$. Show that the boundary value problem

$$\begin{cases} u''(x) + 2u'(x) + u(x) + \lambda \sin^2(u(x)) = f(x), \; x \in [0,1] \\ \\ u(0) = u(1) = 0 \end{cases}$$

has a unique solution $u \in C^2([0,1])$.

(4p)

2. For $f \in L^2(\mathbb{R})$ set

$$Tf(x) = \int_{-\infty}^{\infty} \frac{1}{2} e^{-|x-t|} f(t)\, dt, \; x \in \mathbb{R}.$$

Show that

(a) $Tf \in L^2(\mathbb{R})$ for $f \in L^2(\mathbb{R})$
(b) $T \in \mathcal{B}(L^2(\mathbb{R}))$ with $\|T\| \leq 1$
(c) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\frac{1}{2} e^{-|x-t|}|^2 \, dt dx = \infty$[1].

(4p)

P.T.O.

[1] This means that T is not a Hilbert-Schmidt operator but this concept was only discussed in class and is not given in the literature for the course.

3. Let H be the Hilbert space and P, Q be two orthogonal projections on H. Show[2] that if $P + Q$ is an orthogonal projection then

$$P(H) \perp Q(H).$$

(4p)

4. State and prove Riesz representation theorem. Show that the assumption of "Hilbert space" in the theorem cannot be relaxed to "inner product space".

(5p)

5. Let H be a Hilbert space and assume that $x_n \rightharpoonup x$ in H. Show that $x_n \to x$ in H iff and only if $\|x_n\| \to \|x_n\|$ in $\mathbb{R}$.

(3p)

6. Assume that T is a mapping on a Banach space X and that T^N is a contraction for some positive integer N. Show that
   (a) T has a unique fixed point
   (b) $(T^n(x))_{n=1}^{\infty}$ converges to the fixed point for each $x \in X$
   (c) T does not need to be continuous, for instance by considering the following example: Set $X = \mathbb{R}$ and

$$T(x) = \begin{cases} 1 & x \text{ rational} \\ 0 & x \text{ irrational} \end{cases}$$

   and $N = 2$.

(5p)

For information on the announcement of results see the course homepage where also solutions to the problems will be presented.

GOOD LUCK! PK

[2] Also the converse is true.