MATEMATIK
Chalmers tekniska högskola och Göteborgs universitet

Functional Analysis ENM, TMA401/ Applied Functional Analysis GU, MMA400,
Date: 2018–10–31 (4 hours)

Aids: Just pen, ruler and eraser.
Teacher on duty: Gustav Lindwall, ankn 5325

Note: Write your name and personal number on the cover.
Write your code on every sheet you hand in.
Only write on one page of each sheet. Do not use red pen.
Do not answer more than one question per page.
State your methodology carefully. Write legibly.
Questions are not numbered by difficulty.
Sort your solutions by the order of the questions.
Mark on the cover the questions you have answered.
Count the number of sheets you hand in and fill in the number on the cover.
To pass requires 10 points.

1. Show that the boundary value problem
$$\begin{cases} u'' + \sin^2(u(x)) = x, \ x \in [0,1] \\ \\ u(0) = u(1) = 0 \end{cases}$$
has a unique solution $u \in C^2([0,1])$.

(4p)

2. For $\mathbb{x} = (x_1, x_2, x_3 \ldots) \in l^2$ set $T(\mathbb{x}) = \mathbb{y} = (y_1, y_2, y_3, \ldots)$ where
$$y_1 = x_1, \ y_n = \frac{1}{2^{n-1}}(x_1 + x_2 + \ldots + x_n) \ \text{ for } n \geq 2.$$
Show that T is a bounded linear operator on l^2. Moreover show that T is not surjective.

(4p)

3. Let $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be defined by $T(f)(x) = f(x+1)$ for all $x \in \mathbb{R}$. Show that
   (a) T is a bounded linear operator on $L^2(\mathbb{R})$. Calculate $\|T\|$.
   (b) T has no eigenvalues.

(1+3p)

4. Prove the following version of Banach's fixed point theorem[1]: Let F be a closed set in a Banach space X. Suppose that $T : F \to F$ and that T^N is a contraction on F for some positive integer N. Show that

   (a) T has a unique fixed point.

   (b) The sequence $(T^n(x_0))_{n=1}^{\infty}$ converges to the unique fixed point for every $x_0 \in F$.

(3+2p)

5. Let E be a Hilbert space.

   (a) Define the notion of weak convergence in E, more precisely, what is meant by
   $$x_n \rightharpoonup x \text{ in } E.$$

   (b) Prove that if $x_n \rightharpoonup x$ in E then
   $$\sup_{n=1,2,3,\ldots} \|x_n\| < \infty.$$

(1+3p)

6. Let T be a bounded linear operator on a complex Hilbert space E. Suppose that $\langle T(x), x\rangle \geq 0$ for all $x \in E$. Show that
$$|\langle T(x), y\rangle|^2 \leq \langle T(x), x\rangle\langle T(y), y\rangle \quad \text{all } x, y \in E.$$

(4p)

For information on the announcement of results see the course homepage where also solutions to the problems will be presented.

GOOD LUCK! PK

[1] If a reference to some other version of Banach's fixed point theorem is made then that theorem should also be proven.