MATEMATIK
Chalmers tekniska högskola och Göteborgs universitet

Functional Analysis ENM, TMA401/ Applied Functional Analysis GU, MMA400,
Date: 2019–01–08 (4 hours)

Aids: Just pen, ruler and eraser.
Teacher on duty: Gustav Lindwall, ankn 5325

Note:	Write your name and personal number on the cover. Write your code on every sheet you hand in. Only write on one page of each sheet. Do not use red pen. Do not answer more than one question per page. State your methodology carefully. Write legibly. Questions are not numbered by difficulty. Sort your solutions by the order of the questions. Mark on the cover the questions you have answered. Count the number of sheets you hand in and fill in the number on the cover. To pass requires 10 points.

1. Show that the boundary value problem[1]

$$\begin{cases} u''(x) + u(x) + \arctan(u(x)) = x, \ x \in [0, \frac{\pi}{2}] \\ u(0) = 0, \ u(\frac{\pi}{2}) = \frac{\pi}{2} \end{cases}$$

has a unique solution $u \in C^2([0, \frac{\pi}{2}])$.

(4p)

2. Let $C([0,1])$ be the vector space equipped with the max-norm. Let T_c be the bounded linear operators on $C([0,1])$ defined by

$$T_c(f)(x) = |x-c| \cdot f(x), \ x \in [0,1],$$

where c is a real number. Show that the range $\mathcal{R}(T_c)$ is closed if and only if $c \notin [0,1]$.

(4p)

3. Let $T \in \mathcal{B}(X,X)$ where X is a Banach space. Show that
   (a) $\Sigma_{n=0}^{\infty} \frac{T^n}{n!}$ converges in $\mathcal{B}(X,X)$.
   (b) If $\|T\| < \ln 2$ then $S_m = \Sigma_{n=0}^{m} \frac{T^n}{n!}$ is invertible in $\mathcal{B}(X,X)$ for $m = 1, 2, 3, \ldots$

(1+3p)

[1] Observe that the boundary conditions are **not** homogeneous.

4. State and prove the Lax-Milgram theorem.

(5p)

5. Let $k : [0,1] \times [0,1] \to \mathbb{R}$ be continuous. Set

$$T(f)(x) = \int_0^1 k(x,y) f(y)\, dy,\ x \in [0,1]$$

for continuous functions f on $[0,1]$. Show that

(a) $T \in \mathcal{B}(C([0,1]), C([0,1]))$ where $C([0,1])$ is equipped with the max-norm.

(b) $\|T\| = \max_{x\in[0,1]} \int_0^1 |k(x,y)|\, dy$

(2+2p)

6. Let $\|\cdot\|$ be a norm on a real[2] vector space X. Show that the following statements are equivalent:

(a) $\|\cdot\|$ is induced[3] by an inner product on X.

(b) $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$ for all $x, y \in X$.

(4p)

For information on the announcement of results see the course homepage where also solutions to the problems will be presented.

GOOD LUCK! PK

[2] The statement is also valid for complex vector spaces.

[3] This means that there exists an inner product $\langle \cdot, \cdot \rangle$ such that $\|x\| = \sqrt{\langle x, x \rangle}$ for all $x \in X$.