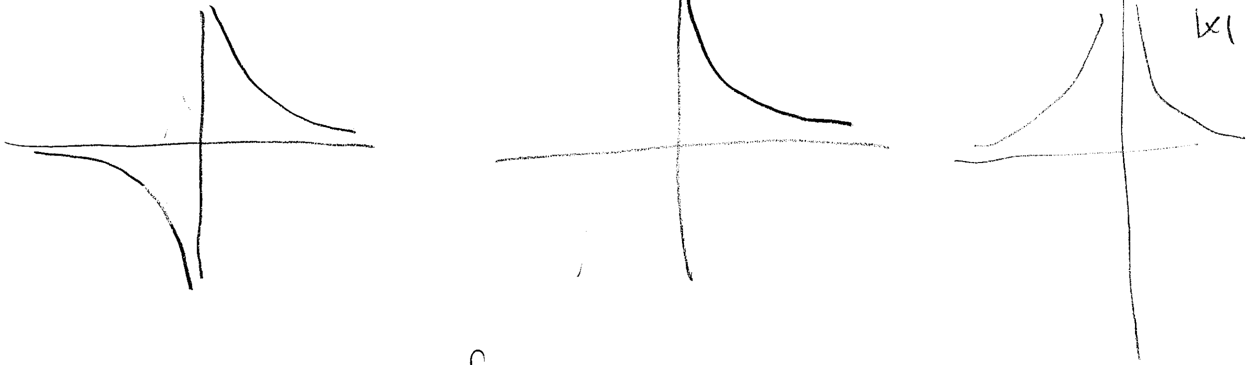
Homogeneous functions

Def a function $f \in C^\infty(\mathbb{R}^n \setminus \{0\})$ is called homogeneous of degree k if
$f(tx) = t^k f(x)$ for $x \neq 0$.

Example $\varphi(x) = \dfrac{1}{x}$ is homogeneous of degree -1

$\varphi(x) = \dfrac{\mathbb{1}_{x>0}}{x}$ — " — -1

$\dfrac{1}{|x|}$



Note that $\langle f, \varphi \rangle = \int_{\mathbb{R}^n} f(x)\varphi(x)\,dx =$

$$= \int f(\alpha x)\alpha^{-k}\varphi(x)\,dx = \int f(y)\,\alpha^{-k}\varphi\left(\frac{y}{\alpha}\right)\frac{dy}{\alpha^n} = \left\langle f, \frac{1}{\alpha^{k+n}}\varphi\left(\frac{\cdot}{\alpha}\right)\right\rangle$$

One can define homogeneity of distributions by this relation.

Ex $\langle \delta, \varphi \rangle = \varphi(0) = \varphi\left(\frac{0}{\alpha}\right)\dfrac{1}{\alpha^{k+n}} \implies k = -n$

So δ is homogeneous of degree $-n$.

Fourier transform of homogen functions (distributions)

Formally, if f is homogeneous of degree k, (in dim 1)

$$\hat{f}(s) = \int_{-\infty}^{\infty} e^{-2\pi i x s} f(x)\,dx$$

$$\hat{f}(ts) = \int_{-\infty}^{\infty} e^{-2\pi i t x s} f(x)\,dx = \int_{-\infty}^{\infty} e^{-2\pi i y s} f\left(\frac{y}{t}\right)\frac{dy}{t}$$

$$= \frac{1}{t^{1+k}} \int_{-\infty}^{\infty} e^{-2\pi i y s} \underbrace{\frac{1}{t^{k}} f\left(\frac{y}{t}\right)}_{= f(y)} dy = \frac{1}{t^{1+k}} \hat{f}(s).$$

So f homogeneous of degree $k \Rightarrow \hat{f}$ hom of degree $\underline{\underline{-1-k}}$

But that this may not be well defined!

__However__ if T hom of degree k

$$\langle \hat{T}, \varphi \rangle = \langle T, \hat{\varphi} \rangle = \langle T, \frac{1}{t^{1+k}} \hat{\varphi}\left(\frac{\cdot}{t}\right) \rangle \quad *$$

$$\frac{1}{t^{1+k}} \hat{\varphi}\left(\frac{s}{t}\right) = \frac{1}{t^{1+k}} \int_{-\infty}^{\infty} e^{-2\pi i x \frac{s}{t}} \varphi(x)\,dx =$$

$$= \int_{-\infty}^{\infty} e^{-2\pi i x y} \varphi(ty) \frac{1}{t^{k}}\,dy =$$

$$\Rightarrow \langle \hat{T}, \varphi \rangle = \langle T, \mathcal{F}\left(\varphi(t\cdot)\frac{1}{t^{k}}\right) \rangle = \langle \hat{T}, t^{-k} \varphi(t\cdot) \rangle$$

$$\langle \hat{T}, \varphi \rangle = \langle \hat{T}, t^{\alpha+n} \varphi(\alpha\cdot) \rangle \quad \text{hom } \alpha$$

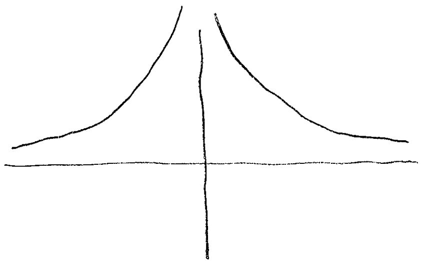
$$\alpha + 1 = -k \Rightarrow \underline{\underline{\alpha = -k-1}}$$

$\Rightarrow$ $\hat{T}$ hom of degree $-k-1$.

Example

$$f(x) = \begin{cases} (-x)^{-1/2} & (x<0) \\ 0 & \text{otherwise} \end{cases}$$

$$f(x) = \frac{1}{2}\left(\frac{1}{|x|^{-1/2}} + \operatorname{sign} x \ \frac{1}{|x|^{-1/2}} \right) = \tfrac{1}{2} f_1(x) + \tfrac{1}{2} f_2(x)$$

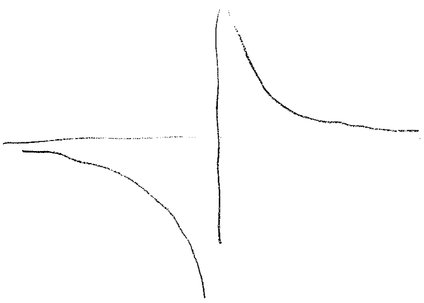


$f_1(x) = |x|^{-1/2}$ even, real, homogeneous of degree $-1/2$

$\Rightarrow \hat{f}_1(s)$ is even, real and homog of degree $-(-\frac{1}{2})-1 = -\frac{1}{2}$

So $\hat{f}_1(s) = \pm c\,|s|^{-1/2}$ + or − ?

we have $\langle \hat{f}_1, e^{-\pi(\cdot)^2} \rangle = \langle f_1, e^{-\pi(\cdot)^2} \rangle > 0 \Rightarrow \hat{f}_1(s) = c\,|s|^{-1/2}$



$f_2(x) = \operatorname{sign} x \ |x|^{-1/2}$
odd, real, hom of degree $-\frac{1}{2}$

$\Rightarrow \hat{f}_2(s)$ is odd, imaginary, hom of degree $-\frac{1}{2}$

so $\hat{f}_2(s) = i\, c \ \operatorname{sign} s \ |s|^{-1/2}$

what is c?

$$2\pi i s \hat{f}_2(s) = \mathcal{F}(f_2')$$

$$\langle 2\pi i s \hat{f}_2, \varphi \rangle = \langle \mathcal{F}(f_2'), \varphi \rangle = \langle f_2', \hat{\varphi} \rangle = -\langle f_2, \hat{\varphi}' \rangle$$

$$\langle -2\pi c\, |\cdot|^{1/2}, e^{-\pi(\cdot)^2} \rangle \qquad = -\langle \operatorname{sign}(\cdot)\,|\cdot|^{-1/2}, 2\pi(\cdot)\, e^{-\pi(\cdot)^2} \rangle$$

$$= -2\pi \langle |\cdot|^{1/2}, e^{-\pi(\cdot)^2} \rangle$$

$\Rightarrow c = 1$

$$\hat{f}(s) = \frac{1}{2}\left(\hat{f}_1(s) - \hat{f}_2(s)\right)$$

$$= \frac{1}{2}\left(|s|^{-1/2} - i \operatorname{sign} s \, |s|^{-1/2}\right) \qquad (\text{for } s \text{ real})$$

For $s > 0$, $\hat{f}(s) = \frac{1-i}{2} \, s^{-1/2} =$

$$\left(\frac{1-i}{2}\right)^2 = \frac{1-2i+i^2}{4} = -\frac{i}{2} = \frac{1}{2i}$$

This can be extended to an analytical

$$= \frac{1}{(2i)^{1/2}} \frac{1}{s^{1/2}} = \frac{1}{(2is)^{1/2}}$$