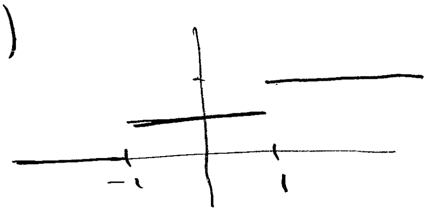
Fourier analys and probability

Let $\xi_1, \xi_2, \ldots$ be i.i.d. random variables,

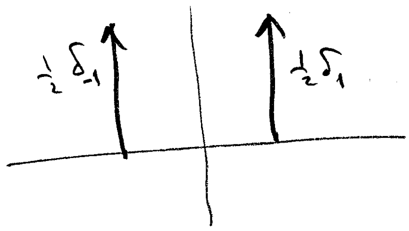
and let $\eta_n = \dfrac{\xi_1 + \ldots + \xi_n}{\sqrt{n}}$.

Assume that $P(\xi_i < x) = \Phi(x)$ (all i)

and that $T = \Phi'$ (in $S'(\mathbb{R})$)



Example: $\Phi = \begin{cases} 0 & \text{when } x < -1 \\ 1/2 & \text{when } -1 < x < 1 \\ 1 & \text{when } x > 1 \end{cases}$



Then $T = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$

This means that $\xi_i = \begin{cases} 1 & \text{with prob } 1/2 \\ -1 & \text{— " —} \end{cases}$

If the law of ξ_i is given by T,
then the law of $\xi_1 + \xi_2$ is given by $T * T$
(when this is well defined ...)

If the law of ξ_i is given by T, then
the law of $a\xi_i$ is given by $\frac{1}{a}T(\frac{\cdot}{a})$

<u>Theorem</u> (the central limit theorem)

Let $\langle T, 1\rangle = 1$, $\langle T, x\rangle = 0$, $\langle T, (\cdot)^2\rangle = \sigma^2 < \infty$.

Then the law of η_n is given by

$$T_n = \underbrace{\sqrt{n}\, T(\sqrt{n}\,\cdot) * \ldots * \sqrt{n}\, T(\sqrt{n}\,\cdot)}_{n \text{ times}},$$

and $T_n \to \dfrac{1}{\sqrt{4\pi\sigma^2}} e^{-\frac{x^2}{4\pi\sigma}}$

when $n \to \infty$.

Proof By the convolution theorem,

$$\hat{T}_n = \mathcal{F}(\sqrt{n}\,T(\sqrt{n}\,\cdot))^n .$$

and

$$\mathcal{F}(\sqrt{n}\,T(\sqrt{n}\,\cdot)) = \hat{T}\left(\frac{\cdot}{\sqrt{n}}\right) = \hat{T}(0) + \hat{T}'(0)\frac{s}{\sqrt{n}} + \frac{1}{2}\hat{T}''(0)\frac{s^2}{n} + O\left(\frac{s^3}{n^{3/2}}\right)$$

Because $\langle T, 1\rangle = 1$, $\hat{T}(0) = 1$,

and $\langle T, (\cdot)\rangle = 0 \Rightarrow \hat{T}'(0) = 0$.

Also $\langle T, (\cdot)^2\rangle = \sigma^2 \Rightarrow \hat{T}''(0) = -4\pi^2\sigma^2$

Then
$$\hat{T}_n(s) = \left(1 - \tfrac{1}{2}4\pi^2\sigma^2\frac{s^2}{n} + O\left(\frac{1}{n^{3/2}}\right)\right)^n$$
$$= \left(1 - \frac{2\pi^2\sigma^2 s^2 + O(1/\sqrt{n})}{n}\right)^n \longrightarrow e^{-2\pi^2\sigma^2 s^2}$$

when $n \to \infty$. And

$$e^{-2\pi^2\sigma^2 s^2} = e^{-\pi(\sqrt{2\pi}\,\sigma s)^2} \subset \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\pi\left(\frac{x}{\sqrt{2\pi}\,\sigma}\right)^2}$$
$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}$$

This then means that

$\eta_n \to \eta$, a random variable with density $\dfrac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2}$.

∎

The law of large numbers

Here we assume that the ξ_i are i.i.d with mean $\bar{\xi}$ and that $E((\xi-\bar{\xi})^2) < \infty$ (this is more than needed). Assume that the law of ξ_i is f

The law of large numbers states that

$$\eta_n = \frac{\xi_1 + \dots + \xi_n}{n} \longrightarrow \delta_{\bar{\xi}}$$

when $n \to \infty$. The proof is similar to the proof of the CLT:

$$\xi_1 + \dots + \xi_n \text{ has Law } \underbrace{f * f * \dots f * f}_{n \text{ times}}$$

and hence that $\frac{\xi_1 + \dots + \xi_n}{n}$ has law $n \cdot \underbrace{f * \dots * f}_{n \text{ times}}(nx) = g_n(x)$

Taking the Fourier transform of g_n, we see that

$$\mathcal{F}(g_n)(s) = \hat{f}(s/n)^n, \quad \text{and because}$$

$$\hat{f}(s) = 1 - 2\pi i \, \hat{f}'(0$$

$$= 1 - 2\pi i \bar{\xi} s + O(s^2)$$

$$\Rightarrow \quad \hat{f}\left(\frac{s}{n}\right)^n = \left(1 - 2\pi i \bar{\xi} \frac{s}{n} + O\left(\frac{s^2}{n^2}\right)\right)^n$$

$$\approx \left(1 - 2\pi i \bar{\xi} \frac{s}{n} \cancel{\frac{s^2}{n^2}} = + O\left(\frac{s^2}{n^2}\right)\right)^n$$

$$= \exp\left(n \log\left(1 - 2\pi i \bar{\xi} \frac{s}{n} + \dots\right)\right)$$

$$= \exp\left(\cancel{n}\left(-2\pi i \bar{\xi} s\right) + O\left(\frac{1}{n}\right)\right) = e^{-2\pi i \bar{\xi} s}$$

$$\Rightarrow \quad f \Rightarrow \delta_{\bar{\xi}}.$$

Bochner's theorem

Def Let $\varphi \in S$ be complex valued.

Then $\varphi \star \varphi(x) = \int \varphi(x+u)\varphi^*(u)\,du$

Recall $\varphi \star \varphi = \varphi * (\varphi^*)^\vee \quad \left(= \varphi * \varphi^*(- \cdot)\right)$

and $\mathcal{F}(\varphi \star \varphi) = |\hat{\varphi}|^2$.

Def $T \in S'$, $\varphi \in S$.

T is called a <u>tempered measure</u> if for all $\varphi \in S$ with compact support (i.e. $\varphi(x) = 0$ for $|x| \geq A$, some A)

$$(*) \qquad |T(\varphi)| \leq C \sup_x |\varphi(x)|$$

Here C may depend on A. If C does not depend on A, we say that T is a bounded measure.

T is called <u>positive</u>, if for all $\varphi \geq 0$, $T(\varphi) \geq 0$.
$(T \geq 0)$

T is called positive definite if for all $\varphi \in S$
$$T(\varphi \star \varphi) \geq 0$$

In $*$: the important thing is that the right hand side does not depend on derivatives of φ.

So δ is a measure, but δ' is <u>not</u>.

Proposition

Let $T \in S'$. Then $\mathcal{F}(T) \geq 0 \Leftrightarrow T$ is positive definite.

Proof For all $\varphi \in S$ the following two statements hold:

1) $\hat{T}(\varphi) \geq 0$ when $\varphi \geq 0 \Leftrightarrow \hat{T}(|\varphi|^2) \geq 0$

2) $T(\varphi \star \varphi) \geq 0$ for all $\varphi \in S \Leftrightarrow \hat{T}(|\hat{\varphi}|^2) \geq 0$ for all $\varphi \in S$.

Proof of 1): $\Rightarrow$ is trivial. To prove $\Leftarrow$, take $0 \leq \psi \in S$, $\psi(x) = 0$ for $|x| > A$.

Let $\psi_n(x) = \left(\psi(x) + e^{-x^2}/n\right)^{\frac{1}{2}}$. ψ_n is strictly positive, and $\psi_n \in S$, and also $\psi_n^2(x) = \psi(x) + e^{-x^2}/n \to \psi(x)$.

Then $0 \leq \hat{T}\left(\psi(x) + \frac{1}{n} e^{-x^2}\right) = \hat{T}(\psi) + \frac{1}{n} \hat{T}(e^{(\cdot)^2})$

$\to \hat{T}(\psi)$;

Therefore $\hat{T}(\psi) \geq 0$

Proof of 2) $\varphi \star \varphi$ is even and

$\varphi \star \varphi = \mathcal{F}\mathcal{F}(\varphi \star \varphi)$. Then $T(\varphi \star \varphi) = T(\mathcal{F}\mathcal{F}(\varphi \star \varphi))$

$= \hat{T}(\mathcal{F}(\varphi \star \varphi)) = \hat{T}(|\hat{\varphi}|^2)$. So $T(\varphi \star \varphi) \geq 0 \Rightarrow \hat{T}(|\hat{\varphi}|^2) \geq 0$.

And this concludes the proof.