The Class $S'(\mathbb{R})$

Notation $\langle \varphi_1, \varphi_2 \rangle = \int_{\mathbb{R}} \varphi_1(x)\, \varphi_2(x)\, dx$

note this is a scalar product only if φ_1, φ_2 are real-valued.

Def A linear mapping $T: S \to \mathbb{C}$ must satisfy:

$$\begin{cases} T(\varphi_1 + \varphi_2) = T\varphi_1 + T\varphi_2 & \varphi_1, \varphi_2 \in S \\ T(\alpha\varphi_1) = \alpha T\varphi_1 & \alpha \in \mathbb{C} \end{cases}$$

Def A linear map $T: S \to \mathbb{C}$

$$\varphi \longmapsto T(\varphi)$$

is called a <u>tempered distribution</u>

if for any sequence $\{\varphi_n\}_{n=1}^{\infty}$, $\varphi_n \in S$,

such that for all $\alpha, \beta \in \mathbb{Z}^+$

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left| |x|^\alpha D^\beta \varphi_n(x) \right| = 0 ,$$

we have

$$\lim_{n\to\infty} T(\varphi_n) = 0$$

Example Take f such that $\frac{f(x)}{(1+x^2)^\alpha}$ is integrable for some $\alpha \geq 0$, and let

$$T(\varphi) = \langle f, \varphi \rangle = \int_{-\infty}^{\infty} f(x)\, \varphi(x)\, dx .$$

Then T is a tempered distribution.

Note We may identify f with T, and sometimes people write $f(\varphi)$ for $T(\varphi)$. This must not be confused with $f(x)$, where we think of $f : \mathbb{R} \to \mathbb{C}$.

Note S is a linear space:

$$\varphi_1, \varphi_2 \in S, \quad \alpha_1, \alpha_2 \in \mathbb{C}$$

$$\Rightarrow \quad \alpha_1 \varphi_1 + \alpha_2 \varphi_2 \in S$$

$$\varphi_1 \equiv 0 \quad \in S$$

To say that $\varphi_n \to \varphi$ in S is equivalent to saying that

$$\lim_{n \to \infty} \sup_{x \in \mathbb{R}} \left| \, |x|^\alpha D^\beta \left(\varphi_n(x) - \varphi(x) \right) \right| = 0 \qquad *$$

The ~~[illegible]~~ family of limits $*$ (indexed by α and β) define a topology in S.

If X and Y are topological spaces, and $f : X \to Y$, we say that f is continuous if

$$f(x_n) \longrightarrow f(x) \quad \text{in } Y$$

whenever $x_n \to x$ in X.

We have $T : S \to \mathbb{C}$. So a tempered distribution is a continuous linear map $S \to \mathbb{C}$.

Exercise

If $f \in C(\mathbb{R})$, $g \in C(\mathbb{R})$

and $f = g$ in S', then $f(x) = g(x)$ for all $x \in \mathbb{R}$

Proof $f = g$ in S' means that for all $\varphi \in S$,

$$\langle f, \varphi \rangle = \langle g, \varphi \rangle \iff \int_{\mathbb{R}} (f(x) - g(x))\, \varphi(x)\, dx = 0.$$

Let $f(x) - g(x) = h(x)$, and assume that there is $x_0 \in \mathbb{R}$ such that $h(x_0) > 0$. Because $h \in C^{*}(\mathbb{R})$, there is an interval $]x_0-\varepsilon, x_0+\varepsilon[$ such that $h(x) > \frac{1}{2} h(x_0)$ in that interval.

Take $\varphi \in S$ such that $\varphi(x) = 0$ when $x \notin]x_0-\varepsilon, x_0+\varepsilon[$ and $\varphi(x) > 0$ when $x \in]x_0-\varepsilon, x_0+\varepsilon[$.

Then

$$\langle h, \varphi \rangle = \int_{-\infty}^{\infty} h(x)\, \varphi(x)\, dx = \int_{x_0-\varepsilon}^{x_0+\varepsilon} h(x)\, \varphi(x) dx$$

$$\geqslant \frac{1}{2} h(x_0) \int_{x_0-\varepsilon}^{x_0+\varepsilon} \varphi(x) dx > 0$$

But this contradicts the statement $\langle f, \varphi \rangle = \langle g, \varphi \rangle$ for all φ.

Example $T : S \to \mathbb{C}$

$$\varphi \mapsto \varphi(a) \qquad a \in \mathbb{R}$$

(i.e., φ is evaluated at the point a)

This is the Dirac "δ-function" at a

1) T is linear

2) Let $\varphi_n \in S$, $\varphi_n \to 0$ in S,

$$\sup_{x \in \mathbb{R}} |x^\alpha D^\beta \varphi_n(x)| \to 0. \qquad (n \to \infty)$$

In particular $\varphi_n(a) \to 0$, so $T(\varphi_n) = \varphi(a) \to 0$.

Note: in this case only φ and no derivatives need to be evaluated

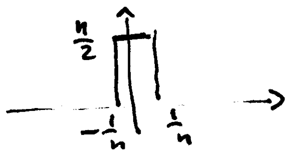
Example Let $f_n(x) = \sqrt{n}\, e^{-n\pi x^2}$ $\quad n = 1, \ldots$

Then $f_n \in S'$ and for all $\varphi \in S$,

$$f_n(\varphi) = \langle f_n, \varphi \rangle \to \varphi(0) = \delta_0(\varphi) \qquad \text{when } n \to \infty$$

We say that $f_n \to \delta_0$ in S'

(other choice of f_n with similar properties:



proof $\quad \langle \varphi_n, \varphi \rangle = \int_{-\infty}^{\infty} \sqrt{n}\, e^{-n\pi x^2} \varphi(x)\, dx$

$$= \int_{-\infty}^{\infty} e^{-\pi x^2} \varphi\left(\frac{y}{\sqrt{n}}\right) dy \to \int_{-\infty}^{\infty} e^{-\pi x^2} \varphi(0)\, dy$$

by uniform convergence (or dominated convergence).

Ex e^x is <u>not</u> a tempered distribution, because it is growing too fast as $x \to \infty$.

Take $\varphi(x) = e^{-\sqrt{1+x^2}} \in S$ (prove that)

Then $\langle e^{(\cdot)}, \varphi \rangle = \int_{-\infty}^{\infty} e^x e^{-\sqrt{1+x^2}}\, dx$,

which is divergent.

Ex Let $\delta_n : \varphi \longmapsto \varphi(n)$, and let

$Ш = \sum_{n=-\infty}^{\infty} \delta_n$, so $\langle Ш, \varphi \rangle = \sum_{n=-\infty}^{\infty} \varphi(n)$

This is a tempered distribution (prove that!)

Next we wish to prove that tempered distributions share many properties with ordinary functions. They can be differentiated, Fourier transformed, etc.

<u>Differentiation of distributions</u>

Let $f \in C^1(\mathbb{R})$, and assume that it is not growing very fast (it may be bounded, for example)

Then $\langle f', \varphi \rangle = \int_{-\infty}^{\infty} f'(x)\varphi(x)\,dx = -\int_{-\infty}^{\infty} f(x)\varphi'(x)\,dx$

which is well defined for all $\varphi \in S$.

<u>Definition</u> Let T be a tempered distribution.

Then we <u>define</u> DT by

$$\langle DT, \varphi \rangle = -\langle T, D\varphi \rangle \quad \text{for all } \varphi \in S.$$

<u>Multiplication by a function</u>

Let $f(x) \in C(\mathbb{R})$ and $g(x) \in C^{\infty}(\mathbb{R})$, and assume that there is an $\alpha \in \mathbb{Z}^{+}$ such that

$\dfrac{|g(x)|}{(1+x^2)^{\alpha}}$ is bounded. Then we have

$$\langle f g, \varphi \rangle = \int_{\mathbb{R}} f(x) g(x) \varphi(x) \, dx = \langle f, g\varphi \rangle$$

<u>Definition</u> Let $T \in S'$, and let g be as above.

Then we <u>define</u> gT by

$\langle gT, \varphi \rangle = \langle T, g\varphi \rangle$. This is <u>ok</u> because

$g\varphi \in S$ if $\varphi \in S$.

<u>Translation</u> Let $f \in C(\mathbb{R})$ and write $f_{\tau}(x) = \tau f(x) = f(x-\tau)$

Then $$\int_{\mathbb{R}} f_{\tau}(x) \varphi(x) \, dx = \int_{\mathbb{R}} f(x-\tau) \varphi(x) \, dx =$$

$$= \int_{\mathbb{R}} f(x) \varphi(x+\tau) \, dx = \langle f, \varphi_{-\tau} \rangle.$$

<u>Def</u> For $T \in S'$, we <u>define</u> T_{τ}

by $\langle T_{\tau}, \varphi \rangle = \langle T, \varphi_{-\tau} \rangle$

But these definitions are only useful if DT, gT and T_τ satisfy some good properties.

<u>Proposition</u> If $T \in S'$, then also DT, gT and T_τ belong to S'.

<u>Proof</u> (for DT)

We have $DT : \varphi \mapsto -\langle T, D\varphi\rangle$.

Then DT is obviously linear (why?).

Take $\{\varphi_n\}$ with $\varphi_n \in S$, and $\varphi_n \to 0$ in S

Then $\sup_x |x^\alpha D^\beta D\varphi_n(x)| = \sup_x |x^\alpha D^{\beta+1} \varphi_n(x)| \to 0$

when $n \to \infty$, and so, because $T \in S'$,

$$\langle T, D\varphi_n\rangle \to 0 \text{ when } n \to \infty.$$

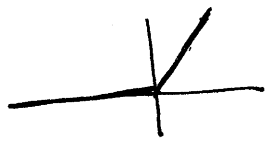
<u>The Structure Theorem</u> Let $T \in S'$.

Then there exist functions $f_j \in C(\mathbb{R})$ such that

$$T = \sum_j D^{\beta_j} f_j$$

That means that any tempered distribution can be written as linear combination of (distributional) derivatives of continuous functions.

<u>Ex</u> Let $f(x) = \begin{cases} x & \text{when } x > 0 \\ 0 & \text{when } x \le 0 \end{cases}$



Then $\langle D^2 f, \varphi\rangle = \int f(x) D^2\varphi(x)\,dx = \int_0^\infty x\, D^2\varphi(x)\,dx$

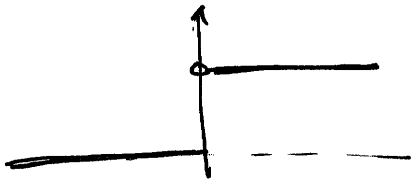
$= -\int_0^\infty D\varphi(x)\,dx = \varphi(0)$, so $D^2 f = \delta_0$.

The proof of the structure theorem is rather difficult.

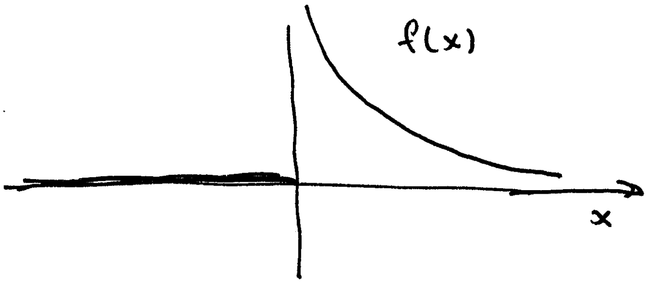
Example

Let $f(x) = \frac{1}{2} x^{-\frac{3}{2}} H(x)$

where $H(x) = \begin{cases} 0 & \text{when } x \le 0 \\ 1 & \text{when } x > 0 \end{cases}$



[note: usually it does not matter if $H(0)$ is defined differently]



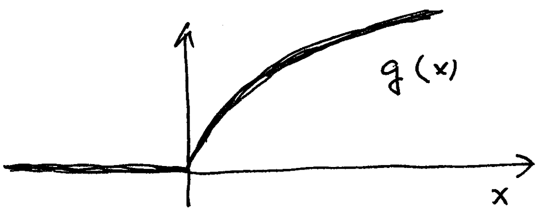
Define a distribution T by

$$\langle T, \varphi \rangle = \lim_{\varepsilon \to 0^+} \left(-\frac{1}{2} \int_\varepsilon^\infty x^{-3/2} \varphi(x)\, dx + \frac{1}{\varepsilon^{1/2}} \varphi(0) \right)$$

Then T is a tempered distribution.

In fact, $T = D^2 g$, where $g(x) = 2 x^{1/2} H(x) \in S'$

(and $\in C(\mathbb{R})$)



But as a function, $f(x)$ is not in S', because

$\int_{-\infty}^{\infty} f(x) \varphi(x)\, dx$ is divergent, in general.

Let $\varphi_n \to 0$ in S.

Then $\langle g, \varphi_n \rangle = \int_0^\infty x^{1/2} \varphi_n(x)\, dx$

$$= \int_0^\infty \frac{x^{1/2}}{1+x^2} (1+x^2)\varphi_n(x)\, dx \to 0 \quad \text{when } n\to\infty$$

because if $\varphi_n \to 0$ in S, so does $(1+x^2)\varphi_n$,

and $\int_0^\infty \frac{x^{1/2}}{1+x^2}\,dx$ is convergent.

Next, $\langle D^2 g, \varphi\rangle = -\langle Dg, D\varphi\rangle = \langle g, D^2\varphi\rangle$

$$= \int_0^\infty 2x^{1/2}\varphi''(x)\,dx = \underbrace{\int_\varepsilon^\infty 2x^{1/2}\varphi''(x)\,dx}_{} + \underbrace{2\int_0^\varepsilon x^{1/2}\varphi''(x)\,dx}_{\to 0 \text{ when } \varepsilon\to 0}$$

$$= -\int_\varepsilon^\infty x^{-1/2}\varphi'(x)\,dx$$

$$= -\left[x^{-1/2}\varphi(x)\right]_\varepsilon^\infty + \int_\varepsilon^\infty -\tfrac{1}{2}x^{-3/2}\varphi(x)\,dx$$

$$= -\frac{1}{2}\int_\varepsilon^\infty x^{-3/2}\varphi(x)\,dx + \varepsilon^{-1/2}\varphi(0) + \underbrace{\varepsilon^{-1/2}(\varphi(\varepsilon)-\varphi(0))}_{\to 0 \text{ when } \varepsilon\to 0.}$$

In conclusion

$$\langle g, D^2\varphi\rangle = -\frac{1}{2}\int_\varepsilon^\infty x^{-3/2}\varphi(x)\,dx + \varepsilon^{-1/2}\varphi(0) + \varepsilon^{-1/2}(\varphi(\varepsilon)-\varphi(0)) + 2\int_0^\varepsilon x^{1/2}\varphi''(x)\,dx$$

$$= \lim_{\varepsilon\to 0}\left(-\frac{1}{2}\int_\varepsilon^\infty x^{-3/2}\varphi(x)\,dx + \varepsilon^{-1/2}\varphi(0)\right).$$

T is called "the finite part" of f.