DFT and the z-transform of $\downarrow 2$ and $\uparrow 2$

Recall $y = (\downarrow 2)\, x = (\ldots\ldots, x_{-2}, x_0, x_2, \ldots)$

$$y_k = x_{2k}$$

$$\Rightarrow \quad Y(z) = \sum_{k=-\infty}^{\infty} x_{2k} z^{-k} = \sum_{k=-\infty}^{\infty} x_{2k} z^{-k}$$

$$= \sum_{k=-\infty}^{\infty} x_{2k} (z^{1/2})^{-2k}$$

$$= \frac{1}{2}\left(\sum_{k=-\infty}^{\infty} x_{2k} (z^{1/2})^{-2k} + \sum_{k=-\infty}^{\infty} x_{2k+1} (z^{1/2})^{-(2k+1)}\right)$$

$$+ \frac{1}{2}\left(\sum_{k=-\infty}^{\infty} x_{2k} (z^{1/2})^{-2k} + \sum_{k=-\infty}^{\infty} x_{2k+1} (-z^{1/2})^{-2k+1}\right)$$

$$= \frac{1}{2}\left(X(z^{1/2}) + X(-z^{1/2})\right)$$

And with $z = e^{i\omega}$, $z^{1/2} = e^{i\omega/2}$, $-z^{1/2} = e^{i(\frac{\omega}{2}+\pi)}$

we get the DFT:

$$Y(\omega) = \frac{1}{2}\left(X\left(\frac{\omega}{2}\right) + X\left(\frac{\omega}{2}+\pi\right)\right)$$

The term $X(\frac{\omega}{2}+\pi)$ is an "alias component",

which can be removed with a "perfect filter" H^*, G^*.

<u>Upsampling</u> $\quad y = (\uparrow 2)\, x = (\ldots\, y_{-1}, 0, y_0, 0, y_1, 0 \ldots)$

Then $$Y(z) = \sum_{k=-\infty}^{\infty} y_k z^{-k} = \sum_{k=-\infty}^{\infty} y_{2k} z^{-2k} = \sum_{k=-\infty}^{\infty} x_k z^{-2k} = X(z^2)$$

and $Y(\omega) = X(2\omega)$.

It is clear then that

$\textcircled{\downarrow 2}\ \textcircled{\uparrow 2}\ x = x$, but $u = \textcircled{\uparrow 2}\ \textcircled{\downarrow 2}\ x \neq x$ in general.

$$U(z) = \sum_{k=-\infty}^{\infty} u_k z^{-k} = \sum_{k=-\infty}^{\infty} x_{2k} z^{-2k}$$

$$= \frac{1}{2}\left[\sum_{k=-\infty}^{\infty} x_{2k} z^{-2k} + \sum_{k=-\infty}^{\infty} x_{2k+1} z^{-(2k+1)} \right]$$

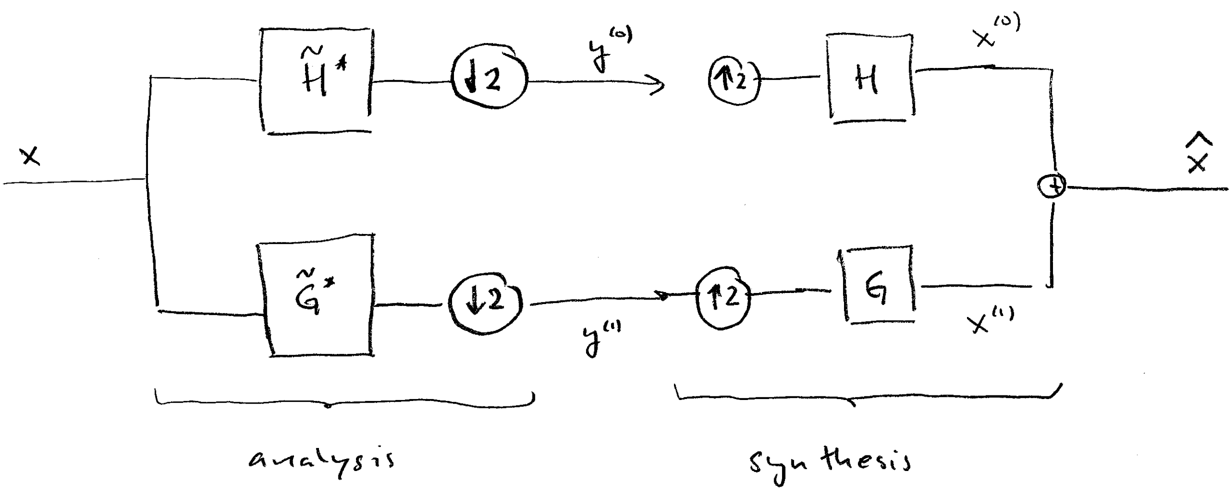
$$+ \frac{1}{2}\left[\sum_{k=-\infty}^{\infty} x_{2k} z^{-2k} - \sum_{k=-\infty}^{\infty} x_{2k+1} z^{-(2k+1)} \right]$$

$$= \frac{1}{2}\left(X(z) + X(-z)\right),$$

which is the same as

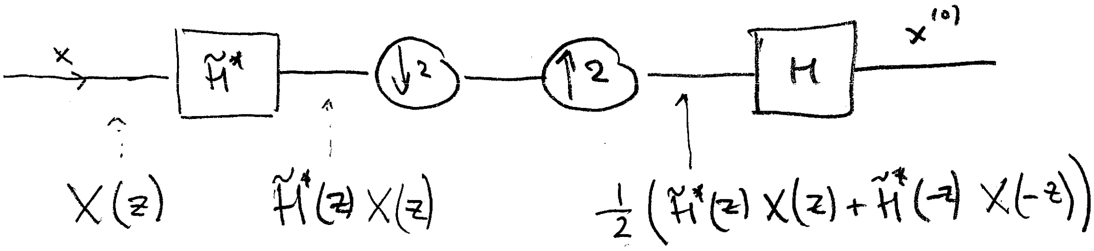
$$U(\omega) = \frac{1}{2}\left(X(\omega) + X(\omega+\pi)\right).$$

Perfect reconstruction



A filter bank. In the analysis part, the signal is split into a high frequency part and a low filter part, wich gives the signals $y^{(0)}$ and $y^{(1)}$.
These then pass the synthesis part to make a reconstruction of the signal. If $\hat{x} = x$, we have achieved "perfect reconstruction"

Expressed in terms of the z-transform, we have, for the high-frequency part.



so that

$$X^{(0)}(z)=\frac{1}{2}H(z)\left[X(z)\tilde{H}^*(z)+X(-z)\tilde{H}^*(-z)\right]$$

and similarly

$$X^{(1)}(z)=\frac{1}{2}G(z)\left[X(z)\tilde{G}^*(z)+X(z)\tilde{G}^*(-z)\right]$$

which gives

$$\hat{X}(z)=\frac{1}{2}\left[H(z)\tilde{H}^*(z)+G(z)\tilde{G}^*(z)\right]X(z)$$
$$+\frac{1}{2}\left[H(z)\tilde{H}^*(-z)+G(z)\tilde{G}^*(-z)\right]X(-z)$$

We therefore achieve perfect reconstruction if and only if the following system is satisfied:

$$\begin{cases} H(z)\tilde{H}^*(z)+G(z)\tilde{G}^*(z)=2 \\ H(z)\tilde{H}^*(-z)+G(z)\tilde{G}^*(-z)=0 \end{cases}.$$

How can we construct such filters?

The "product filters" are obtained by the ansatz

$$G(z)=-z^{-L}\tilde{H}^*(-z) \qquad L \text{ odd}$$
$$\tilde{G}(z)=-z^{-L}H^*(-z)$$

Recall that the $*$-marked filters are time reversed, so

$$G^*(z) = \sum_{k=-\infty}^{\infty} g_k^* z^{-k} = \sum_{k=-\infty}^{\infty} g_k \left(\frac{1}{z}\right)^{-k} = G\left(\frac{1}{z}\right)$$

and therefore

$$\tilde{G}^*(-z) = -\left(-\frac{1}{z}\right)^{-L} H^*\left(\frac{1}{z}\right) = \left(\frac{1}{z}\right)^{-L} H(z)$$

Therefore

$$H(z)\tilde{H}^*(-z) + G(z)\tilde{G}^*(-z) = H(z)\tilde{H}(-z) - z^{-L} H^*(-z)\left(\frac{1}{z}\right)^{L} H(z) = 0$$

and

$$H(z)\tilde{H}^*(z) + G(z)\tilde{G}^*(z) =$$

$$= H(z)\tilde{H}^*(z) - z^{-L}\tilde{H}^*(-z) \cdot \left(\frac{1}{-z}\right)^{-L} H(-z)$$

$$= H(z)\tilde{H}^*(z) + \tilde{H}^*(-z)H(-z).$$

The condition for perfect reconstruction is now given in terms of H and $\tilde{H}$:

$$\boxed{H(z)\tilde{H}^*(z) + \tilde{H}^*(-z)\,H(-z) = 2}$$

and a <u>product filter</u> is defined by

$$P(z) = H(z)\tilde{H}^*(z).$$

The perfect reconstruction condition becomes

$$P(z) + P(-z) = 2, \quad \text{i.e.}$$

$$2p_0 + 2\sum_{k=-\infty}^{\infty} p_{2k} z^{-2n} = 2.$$

It follows that $p_0 = 1$, $p_{2n} = 0$ $(n \neq 0)$, and p_{2k+1} can be chosen arbitrarily.

Given $P(z)$ we can choose $H(z)$ and $\tilde{H}^*(z)$, but this may be done in many different ways.

Orthogonal Filter banks

Filters that satisfy $H(z) = \tilde{H}(z)$, $G(z) = \tilde{G}(z)$

are called orthogonal. Then

$$P(z) = H(z)\tilde{H}^*(z) = H(z)\,H(\tfrac{1}{z}).$$

In the DFT- version:

$$P(\omega) = H(\omega)\overline{H(\omega)} = |H(\omega)|^2 \geq 0.$$

Then $P(\omega)$ is even, and real valued

which implies that the coefficients are symmetric,

and perfect reconstruction follows from

$$|H(\omega)|^2 + |H(\omega+\pi)|^2 = 2.$$

Why are these called orthogonal?

Recall that in the example, we had

$$\varphi_k = \begin{cases} \frac{1}{\sqrt{2}} & k=0,1 \\ 0 & \text{otherwise} \end{cases} \qquad \psi_k = \begin{cases} 1/\sqrt{2} & k=0 \\ -1/\sqrt{2} & k=1 \\ 0 & \text{otherwise} \end{cases}$$

and we constructed basis functions

$$\varphi_k^{(2n)} = \varphi_{k-2n}, \quad \varphi_k^{(2n+1)} = \psi_{k-2n}, \quad \text{and then}$$

$$y_n^{(0)} = \langle x, \varphi^{(2n)} \rangle \qquad y_n^{(1)} = \langle x, \varphi^{(2n+1)} \rangle, \quad \text{and in}$$

the end:

$$x = \sum \langle x, \varphi^{(2n)} \rangle \varphi^{(2n)} + \sum \langle x, \varphi^{(2n+1)} \rangle \varphi^{(2n+1)}.$$

Biorthogonal bases

Sometimes it is advantageous to work with "biorthogonal" bases, which corresponds to expansions of the form

$$x = \sum_n \langle x, \tilde{\varphi}^{(2n)} \rangle \varphi^{(2n)} + \sum_n \langle x, \tilde{\varphi}^{(2n+1)} \rangle \varphi^{(2n+1)}.$$

The basis functions must then satisfy

$$\langle \tilde{\varphi}^{(2k)}, \varphi^{(2n)} \rangle = \delta_{k,n}$$

$$\langle \tilde{\varphi}^{(2k+1)}, \varphi^{(2n+1)} \rangle = \delta_{k,n}$$

$$\langle \tilde{\varphi}^{(2k)}, \varphi^{(2n+1)} \rangle = \langle \tilde{\varphi}^{(2k+1)}, \varphi^{(2n)} \rangle = 0.$$

We will return to that when discussing multi resolution analysis.

Design of filter banks

This can be done in three steps:

1) Find $P(z)$ that satisfies the condition for perfect reconstruction

2) Factorize $P(z)$:
$$P(z) = H(z)\,\tilde{H}^{*}(z).$$

3) Define the high pass filters by

$$G(z) = -z^{-L}\,\tilde{H}^{*}(-z)$$
$$\tilde{G}(z) = -z^{-L}\,H^{*}(-z) \quad , \text{ where } L \text{ is odd.}$$

Example

The Haar basis: $P(z) = \frac{1}{2}(z + 2 + z^{-1})$

$$= \frac{1}{\sqrt{2}}(z+1) \frac{1}{\sqrt{2}}\left(\frac{1}{z}+1\right)$$

Example (Ingrid Daubechie)

$$P(z) = \left(\frac{1+z}{2}\right)^N \left(\frac{1+z^{-1}}{2}\right)^N Q_N(z),$$

where $Q_N(z)$ is symmetric:

$$Q_N(z) = a_{N-1} z^{N-1} + \dots + a_1 z^1 + a_0 + a_1 z^{-1} + \dots + a_{N-1} z^{1-N}$$

$N=1$ then gives the Haar wavelet,

$N=2$:

$$P(z) = \left(\frac{1+z}{2}\right)^2 \left(\frac{1+z^{-1}}{2}\right)^2 Q_2(z),$$

$$Q_2(z) = a_1 z + a_0 + a_1 z^{-1}.$$

Wavelets

Recall the Fourier transform (NB: definition)

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt = \langle f, e^{i\omega\cdot}\rangle$$

Def $\quad \langle f, g\rangle = \int f(t)\overline{g(t)}\,dt$

The Fourier transform gives a decomposition in pure frequencies: $e^{i\omega t}$,

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \langle f, e^{i\omega\cdot}\rangle e^{i\omega t}\,d\omega.$$

The wavelet decomposition is different.

Let $\psi(t)$ be a given function, and let

$$\psi_{j,k}(t) := 2^{j/2}\psi(\underset{\uparrow\,\text{dilation}}{2^j} t - \underset{\text{translation}}{k}).$$

Under suitable conditions,

$$f(t) = \sum_{j,k=-\infty}^{\infty} \langle f, \psi_{j,k}\rangle \psi_{j,k}(t).$$

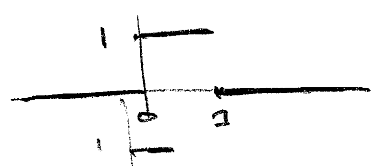
<u>Example</u> The Haar wavelet.

Let $\quad \varphi(t) = \varphi(2t) + \varphi(2t-1) \quad \leftarrow$ need to check that it is possible

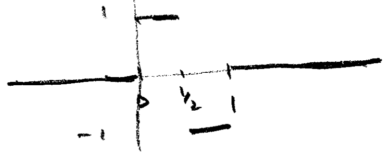
$\quad\quad\ \psi(t) = \varphi(2t) - \varphi(2t-1) \quad \leftarrow$ can always be achieved.

φ is called a scaling function and ψ a wavelet.

The Haar scaling function is



and the wavelet is

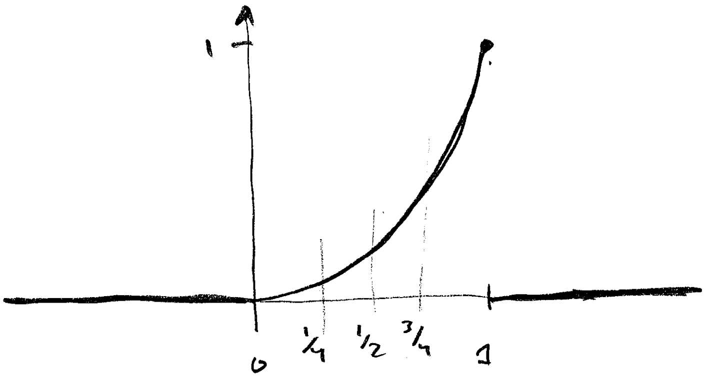


Note that $\langle \varphi, \psi \rangle = 0$,

$$\langle \varphi(\cdot - k), \varphi(\cdot - n) \rangle = 0 \quad (n \neq k)$$

$$\langle \psi(\cdot - k), \psi(\cdot - n) \rangle = 0 \quad (n \neq k)$$

As an example, let $f(t) = \begin{cases} t^2 & 0<t<1 \\ 0 & \text{elsewhere} \end{cases}$



The function $f(t)$ can be approximated by step functions, which is the mean of f in each interval

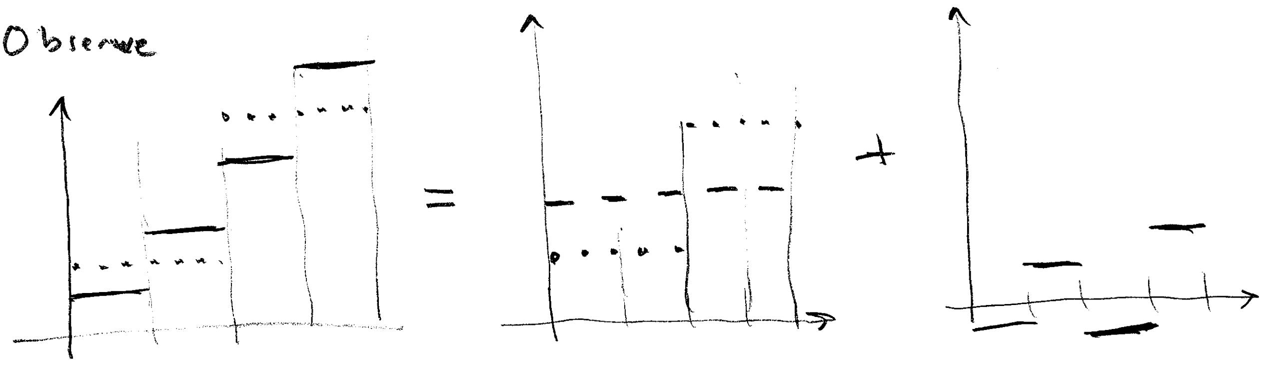
For example, in the interval $[\frac{1}{4}, \frac{1}{2}]$

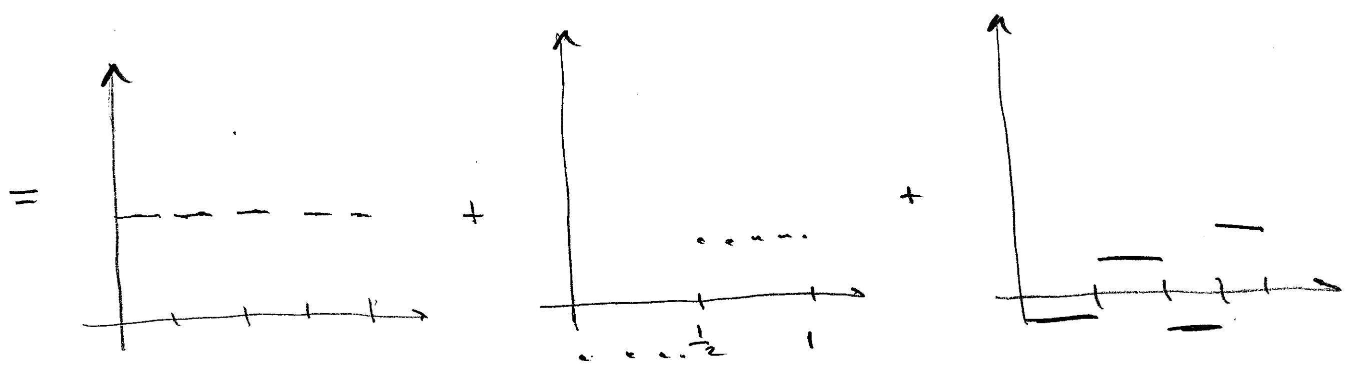
we have

$$\langle f, 2\varphi(2^2 \cdot -1) \rangle = \int_0^1 t^2\, 2\varphi(2^2 t - 1)\, dt$$

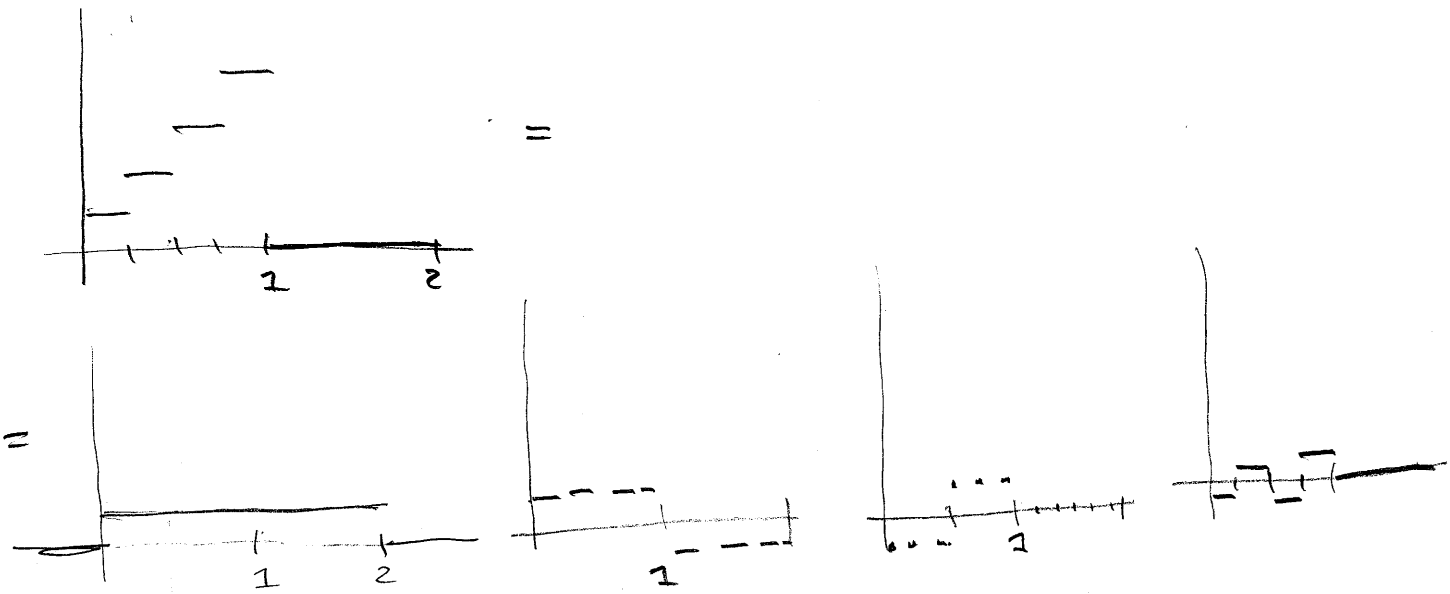
$$= \int_{1/4}^{1/2} t^2\, 2\, dt \approx 0.073.$$

Observe





This is the basic idea behind wavelets:
we can express the signal as an average plus finer and finer details. Note that we can continue:



$2^{-1/2}\varphi\left(\frac{t}{2}\right)$. Note that $\langle f, 2^{j/2}\varphi(2^j\cdot - k)\rangle \to 0$ when $j \to -\infty$ (i.e. taking the average over larger and larger intervals.

Multi resolution analysis

The Hilbertspace $L^2(\mathbb{R})$

$$L^2(\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{C} : \int |f(t)|^2 dt < \infty \}$$

(for a proper definition, we need the concept of measurability).

The <u>norm</u> of f is defined as

$$\|f\| = \left(\int_{\mathbb{R}} |f(t)|^2 dt \right)^{1/2}$$

Properties of a norm: $\|f\| \geq 0$, $\|f\| = 0 \Leftrightarrow f = 0$

$\|cf\| = |c| \, \|f\| \qquad c \in \mathbb{C}$

$\|f+g\| \leq \|f\| + \|g\|$.

In a Hilbert space, the norm is defined via the scalar product: $\|f\| = \langle f, f \rangle^{1/2}$

Recall the Parseval formula: $\int_{\mathbb{R}} |\hat{f}(\omega)|^2 d\omega = 2\pi \int_{\mathbb{R}} |f(t)|^2 dt$.

Closed subspaces and projections

A linear subspace $V \subset L^2(\mathbb{R})$ is <u>closed</u> if $f_n \in V$ $(n = 1 \ldots \infty)$ and $f_n \to f$ in $L^2(\mathbb{R})$ implies that $f \in V$.

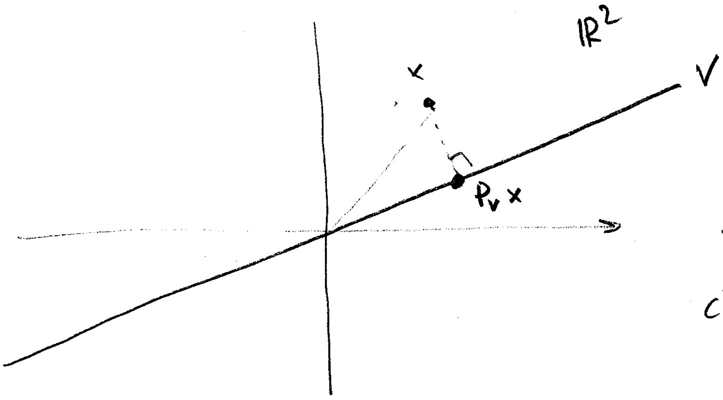
<u>Example</u> $\{ f \in L^2(\mathbb{R}), \; f(t) = 0 \text{ when } t < 0 \}$

$$V_{BL} = \{ f \in L^2(\mathbb{R}) : |\hat{f}(\omega)| = 0 \text{ when } |\omega| > a \}$$

<u>Def</u> A linear operator $P : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is called a <u>projection</u> if

$$P(Pf) = Pf \quad \text{for all } f \in L^2(\mathbb{R}).$$

<u>Def</u> The orthogonal projection of f onto the closed subspace V is the unique $w \in V$ such that $\|f-w\| \leq \|f-v\|$ for all $v \in V$. This projection is denoted P_V.



Here is an example of a subspace of $\mathbb{R}^2$. It is clear that

$$P_v(P_v x) = P_v x.$$

Note that $\langle f - P_v f, v \rangle = 0$ for all $v \in V$.

<u>Bases</u> A family of functions $\{\varphi_k\}_{k \in \mathbb{Z}}$ is a <u>basis</u> for $V \subset L^2(\mathbb{R})$ if any $f \in V$ can be written (in a unique way) as

$$f = \sum_k c_k \varphi_k,$$

where $c_k \in \mathbb{C}$.