Fourier and Wavelet Analysis

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-2\pi i x \xi} f(x)\, dx$$

[this formula works equally well in $\mathbb{R}^n$, if $\int_{-\infty}^{\infty}$ is replaced by $\int_{\mathbb{R}^n}$, and $x\xi$ denotes the scalar product $\langle x, \xi \rangle$]

The course

- The Fourier transform
- Distribution theory
- Transforms related to the Fourier transform (Radon, Hankel, Z ...)
- The Wavelet transforms
- Multi resolution analysis
- Discret transforms, sampling
- Applications (three lab assignments)

Notation

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i x \xi} f(x)\, dx$$

$$\hat{f}(\xi) = (\mathcal{F} f)(\xi)$$

$$f(x) \supset \hat{f}(\xi)$$

Basic properties

Linearity: $\quad f + g \supset \hat{f} + \hat{g}$

$\qquad \alpha f \supset \alpha \hat{f} \qquad (\alpha \in \mathbb{R}, \mathbb{C})$

Scaling: $\quad f \supset \hat{f} \iff \frac{1}{a} f\left(\frac{x}{a}\right) \supset \hat{f}(a\xi) \qquad (a>0)$

(better notation: $\frac{1}{a} f\left(\frac{\cdot}{a}\right) \supset \hat{f}(a\xi)$)

"if f is nice, then $\hat{f}$ is nice"
(this will be made precise later)

Fourier transform in L^2

$$L^2(\mathbb{R}) = \left\{ f, \text{ measurable, such that } \int_{\mathbb{R}} |f(x)|^2 dx < \infty \right\}$$

Scalar product: $\quad \langle f, g \rangle = \int_{\mathbb{R}} f(x) \overline{g(x)}\, dx$

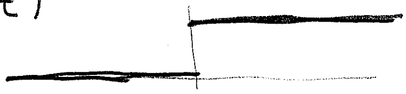
$$\int_{\mathbb{R}} |f(x)|^2 dx = \int_{\mathbb{R}} |\hat{f}(\xi)|^2 d\xi$$

By nice we will mean: Schwartz class $\mathcal{S}$

Often: $f(t)$, a signal

$f(t) = \sin(\omega t)$

$f(t) = H(t)$



"$f(t) = \delta(t)$"

the Dirac "δ-function"

Those are <u>not</u> in L^2, nor in $\mathcal{S}$.

<u>Fundamental</u> The Fourier inversion formula

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i x \xi} f(x)\,dx$$

$$\Longleftrightarrow \quad f(x) = \int_{\mathbb{R}} e^{2\pi i x \xi} \hat{f}(\xi)\,d\xi$$

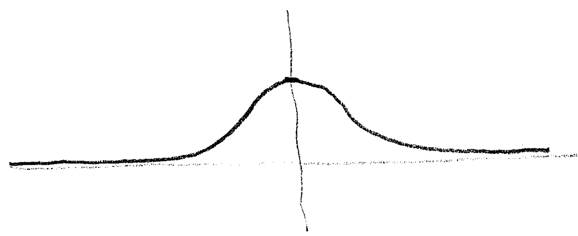
[which means that

$$f(x) = \int_{\mathbb{R}} e^{2\pi i x \xi} \underbrace{\left[\int_{\mathbb{R}} e^{-2\pi i y \xi} f(y)\,dy \right]}_{\hat{f}(\xi)} d\xi$$

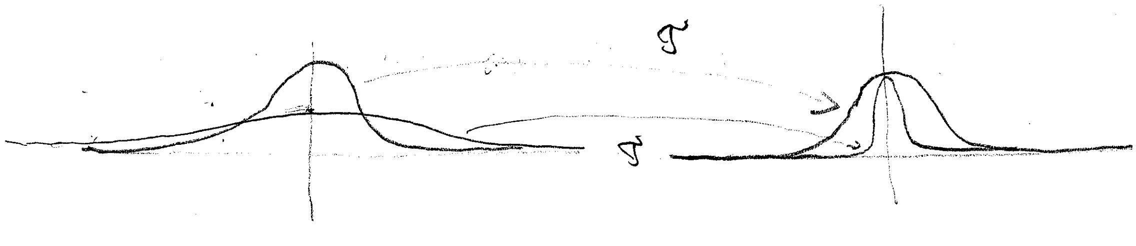
]

<u>Example</u> (important)

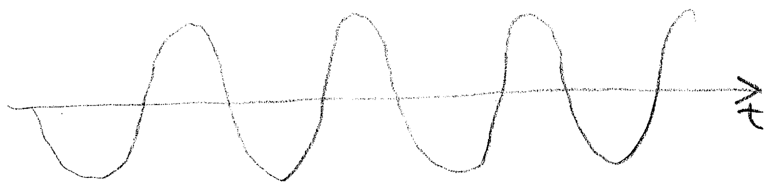
$$\mathcal{F}\left(e^{-\pi x^2}\right) = e^{-\pi \xi^2}$$



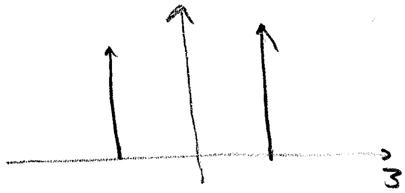
(scaling: $\frac{1}{2} e^{-\pi\left(\frac{x}{2}\right)^2} \supset e^{-\pi(2\xi)^2}$



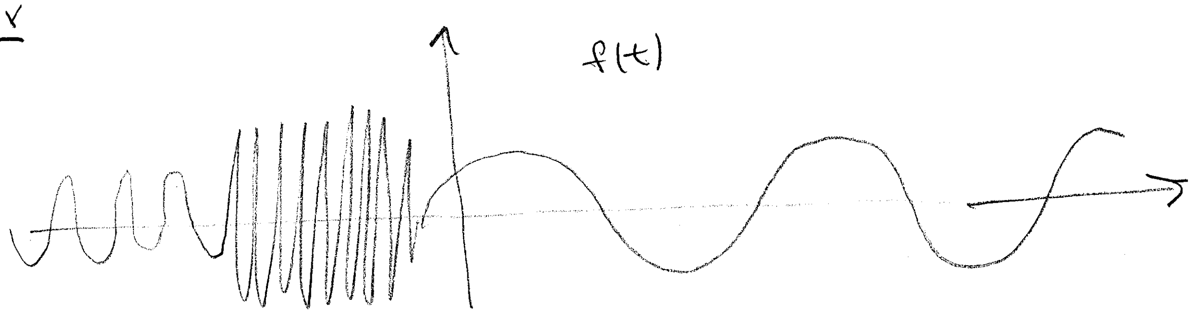
Ex $f(t) = \sin(\omega t)$



$$\hat{f}(\xi) = \frac{1}{2}\left(\delta_{-\omega/2\pi} + \delta_{\omega/2\pi}\right)$$



Ex



The Fourier transform does not distinguish between what happened a long time ago, now, and in the future.

One solution: Wavelets (another: windowed Fourier transform; STFT)

<u>Our other transforms</u>:

Hankel: $$\hat{f}(\xi_1, \xi_2) = \iint_{\mathbb{R}^2} e^{-2\pi i(\xi_1 x_1 + \xi_2 x_2)} f(x_1, x_2)\, dx_1 dx_2$$

$x_1 = r\cos\theta$
$x_2 = r\sin\theta$

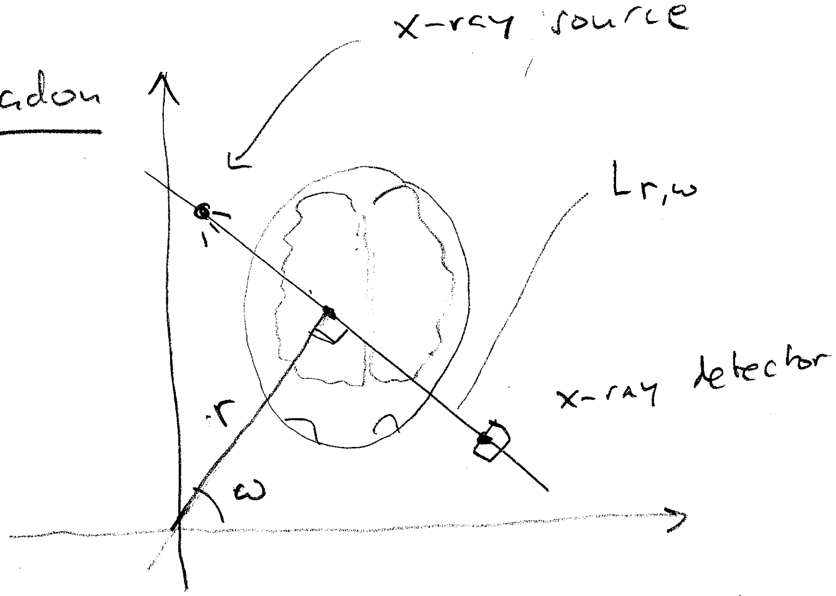
Let $f(x_1, x_2) = F\left(\sqrt{x_1^2 + x_2^2}\right) = F(r)$

Then $\hat{f}(\xi_1, \xi_2) = \tilde{F}\left(\sqrt{\xi_1^2 + \xi_2^2}\right) = \tilde{F}(\rho)$

$\xi_1 = \rho\cos\alpha$
$\xi_2 = \rho\sin\alpha$

$F(r) \longmapsto \tilde{F}(\rho)$ is the Hankel transform.

Radon



If $f(x_1, x_2)$ is the density of the head at point (x_1, x_2), then

$$\int_{L_{r,\omega}} f(x_1, x_2)\, dl$$ gives a measure of the damping along $L_{r,\omega}$. This can be measured.

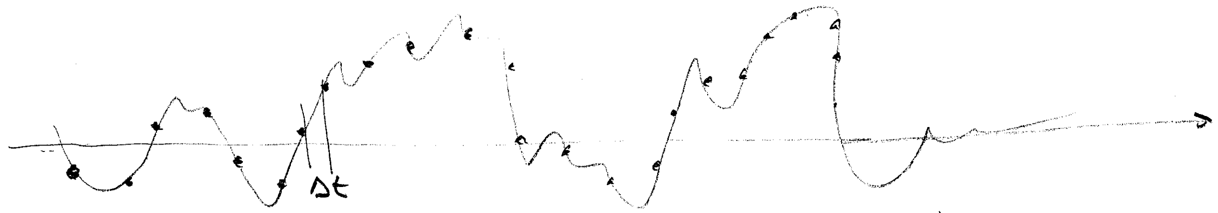
Notation: $$R_\omega f(r) = \int_{L_{r,\omega}} f(x_1, x_2)\, dl,$$

the Radon transform.

Important because there is a formula for recovering f from $R_\omega f(r)$,

(computer tomography).

Discrete transforms, sampling, filters



$$f(t) \longrightarrow \{f_n\}_{n=-\infty}^{\infty} = \{f(n\,\Delta t)\}_{n=-\infty}^{\infty}$$

$\{f_n\}$ is a discrete signal.

How often do we need to sample?

How do we construct digital filters?

Low pass, high pass ....

Fast Fourier Transform

What functions do have a Fourier-transform:

$L^2(\mathbb{R})$

S (smooth functions with rapid decay)

$L^1(\mathbb{R}) = \{ f : \int_{\mathbb{R}} |f(x)|\,dx < \infty \}$

<u>Note</u> $f(x) \in L^1(\mathbb{R}) \Rightarrow |\hat{f}(\xi)| = \left| \int_{\mathbb{R}} e^{-i2\pi\xi x} f(x)\,dx \right|$

$$\leq \int_{\mathbb{R}} |e^{-2\pi i x\xi} f(x)|\,dx = \int_{\mathbb{R}} |f(x)|\,dx < \infty$$

($f \in L^1 \Rightarrow \hat{f} \in L^\infty$)

Hausdorff Young's inequality $f \in L^p$ ($1 \leq p \leq 2$)

$$\Rightarrow \hat{f} \in L^q \qquad q = \frac{1}{1-\frac{1}{p}}$$

<u>Convolution theorem</u>

Suppose that $f(x) \supset \hat{f}(\xi)$, $g(x) \supset \hat{g}(\xi)$.

Then

$$f * g(x) \equiv \int_{\mathbb{R}} f(x-y)\,g(y)\,dy \supset \hat{f}(\xi)\hat{g}(\xi)$$

Other properties of convolution:

commutative: $f * g = g * f$

associative: $f * (g * h) = (f * g) * h$

distributive: $f * (g+h) = f * g + f * h$