FFT by decimation in time

1) Split f into $f_O = \{f_1, f_3, \ldots, f_{2N-1}\}$ (odd)

$f_E = \{f_0, f_2, \ldots, f_{2N-2}\}$ (even)

2) compute DFT of f_O and f_E:

$f_O \supset F_O$, $f_E \supset F_E$

3) then $f \supset \frac{1}{2} R_2 F_E + \mathcal{E}_1 \frac{1}{2} R_2 F_O$

Let $\chi(N)$ = cost of computing DFT of a sequence of length N.

Then $\chi(2N)$ = cost of splitting ...> $O(N)$

+ cost of computing $\frac{1}{2} R_2$ ...> $O(N)$

+ $\chi(N)$

+ cost of computing $\mathcal{E}_1$ ...> $O(N)$

+ cost of $\frac{1}{2} R_2$ ...> $O(N)$

+ $\chi(N)$

+ cost of adding ...> $O(N)$

Hence the total cost is

$$\chi(2N) = O(N) + 2\chi(N)$$

If $N = 2^m$

$$\chi(N) = O(2^m) + 2\chi(2^{m-1})$$
$$= O(2^m) + 2(O(2^{m-1}) + 2\chi(2^{m-2}))$$

$$
\begin{aligned}
&= 2\,\mathcal{O}(2^m) + 4\,X(2^{m-2}) \\
&= 2\,\mathcal{O}(2^m) + 4\left(\mathcal{O}(2^{m-2}) + 2\,X(2^{m-3})\right) \\
&= 2\,\mathcal{O}(2^m) + \mathcal{O}(2^m) + 2^3\,X(2^{m-3}) \\
&= 3\,\mathcal{O}(2^m) + 2^3\,X(2^{m-3}) \\
&= \dots = m\,\mathcal{O}(2^m) + 2^m\,X(1) \\
&= m\,\mathcal{O}(N) + N \quad \Rightarrow \quad \text{total cost} = \underline{\mathcal{O}(N \log_2 N)}
\end{aligned}
$$

"Decimation in frequency"

$$
\begin{tikzcd}
\{f_0, f_1, f_2, f_3\} \arrow[r, "\mathcal{F}"] \arrow[d, "\mathcal{E}_1"'] & \{F_0, \dots, F_3\} \arrow[d, "T_{-1}"] \\
\{f_0, \omega f_1, \dots, \omega^3 f_3\} \arrow[r, "\mathcal{F}"] & \{F_1, \dots, F_0\}
\end{tikzcd}
$$

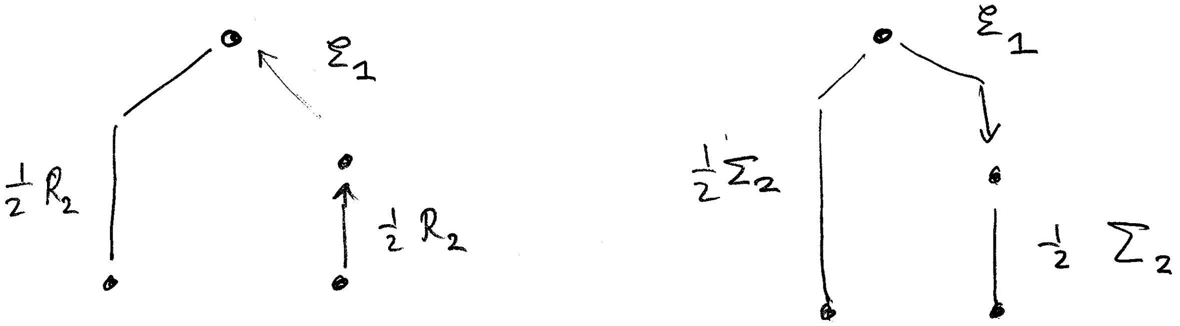
$$
\begin{tikzcd}
\{f_0, \dots, f_3\} \arrow[r, "\mathcal{F}"] \arrow[d, "\frac{1}{2}\Sigma_2"'] & \{F_0, \dots, F_3\} \arrow[d, "\Xi_2"] \\
\frac{1}{2}\{f_0 + f_2, f_1 + f_3\} \arrow[r, "\mathcal{F}"] & \{F_0, F_2\}
\end{tikzcd}
$$

Σ_2 : summation $\quad (\Sigma_2 f)(\tau) = f(\tau) + f(\tau + \frac{N}{2})$

Ξ : down sampling

$$
\begin{tikzcd}
 & \{f_0, \dots, f_3\} \arrow[d, "\mathfrak{F}"] \arrow[ddd, bend right, "\frac{1}{2}\Sigma_2"'] \arrow[ddrr, "\mathcal{E}_1"] & & & \\
 & \{F_0, \dots, F_3\} \arrow[dd, "\Xi_2"'] \arrow[dr, "\tau_{-1}"] & & & \\
 & & \{F_1, F_2 F_3, F_0\} \arrow[d, "\Xi_2"'] & \{\ \} \arrow[l, "\mathfrak{F}"'] \arrow[d, "\frac{1}{2}\Sigma_2"] & \\
\{\ \} \arrow[r, "\mathfrak{F}"] & \{F_0, F_2\} & \{F_1, F_3\} \arrow[dr, "\tau_{-1}"] \arrow[dd, "\Xi_2"'] & \{\ \} \arrow[l, "\mathfrak{F}"'] \arrow[dr, "\mathcal{E}_4"] & \\
 & & & \{F_3, F_1\} \arrow[d, dash, "\Xi_2"'] & \{\ ,\ \} \arrow[l, "\mathfrak{F}"'] \arrow[d, "\frac{1}{2}\Sigma_2"] \\
 & & \{F_1\} & \{F_3\} & \{\ \} \arrow[l, "\mathfrak{F}"']
\end{tikzcd}
$$

Tree structures:



The Fourier transform (again)

$$f \supset \hat{f} \qquad \hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-2\pi i x \xi} f(x)\,dx$$

Autocorrelation

Def $$f \star f = \int_{-\infty}^{\infty} f(u) f(u-x)\,du = \int_{-\infty}^{\infty} f(u) f(u+x)\,du$$

Normalized

$$\gamma(x) = \frac{f \star f\,(x)}{\int_{-\infty}^{\infty} |f(u)|^2\,du}$$

Then $\gamma(x) \leq 1$, $\gamma(0) = 1$, because

$f \star f\,(x)$ has a maximum at $x=0$

and this is because $ab \leq \frac{1}{2}(|a|^2 + |b|^2)$,

so $$\int_{-\infty}^{\infty} f(u) f(x+u)\,du \leq \frac{1}{2}\int_{-\infty}^{\infty} \left(f(u)^2 + f(x+u)^2\right) du$$

$$= \int_{-\infty}^{\infty} f(u)^2\,du$$

Recall that $f: \mathbb{R} \to \mathbb{C}$, $f \supset \hat{f}$

implies

$$\widehat{f^*} = \hat{f}(-\cdot)^* \qquad \left(\begin{array}{l} z = x+iy \\ z^* = x-iy \end{array} \right)$$

and $$\mathcal{F}(f^*(-\cdot)) = \hat{f}^*.$$

Therefore $$f * f^*(-\cdot) = \hat{f}\hat{f}^* = |\hat{f}|^2$$

And

$$f * f^*(-\cdot)\,(x) =$$

$$= \int_{-\infty}^{\infty} f(y)\, f(-(x-y))^* \,dy = \int_{-\infty}^{\infty} f(y)\, f(y-x)^* \,dy$$

$$= f \star f^*, \quad \text{and if } f \text{ is real,}$$

we find

$$f \star f = |\hat{f}|^2$$

Def $|\hat{f}|^2$ is called "the energy density"

Some relations

1) $\int f(x)\,dx = \hat{f}(0)$

2) $\int x f(x)\,dx = \dfrac{\hat{f}'(0)}{-2\pi i}$

Def $\langle x \rangle = \dfrac{\int x f(x)\,dx}{\int f(x)\,dx} = -\dfrac{1}{2\pi i} \dfrac{\hat{f}'(0)}{\hat{f}(0)}$

Def $\int x^2 f(x)\,dx = -\dfrac{1}{4\pi^2} \hat{f}''(0)$ (moment of inertia)

$\langle x^2 \rangle = \int x^2 f(x)\,dx \Big/ \int f(x)\,dx = -\dfrac{1}{4\pi^2} \dfrac{\hat{f}''(0)}{\hat{f}(0)}$

Def Variance $\sigma^2 = \langle (x - \langle x \rangle)^2 \rangle$

Note if $f(x) = 0$ for $|x| > a$ and $f \in L^1$, then $\hat{f}(\xi) \in C^\infty$, because

$$|D^k \hat{f}(\xi)| = \left| \int_{-\infty}^{\infty} (-2\pi i x)^k f(x)\,dx \right| \leq (2\pi a)^k \int_{-\infty}^{\infty} |f(x)|\,dx < \infty.$$

The uncertainty relation

The Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H}(\psi)$$ $\psi = \psi(r,t)$, the wave function

For a single particle:

$$i\hbar \frac{\partial \psi}{\partial t}(r,t) = -\frac{\hbar^2}{2m} \Delta \psi(r,t) + V(r)\,\psi(r,t)$$

potential energy

kinetic energy ($\nabla\psi$ corresponds to momentum)

$\hat{p} = -i\hbar \frac{\partial}{\partial x}$ (momentum)

x (position)

Heisenberg: you cannot know velocity and position of a particle exactly at one time.

Prel

1) $$|f(x)| = \left|\int_{\mathbb{R}} \hat{f}(\xi)\,d\xi\right| \le \int_{\mathbb{R}} |\hat{f}(\xi)|\,d\xi$$

2) (Cauchy Schwarz)

$$\left|\int_{\mathbb{R}} (f g^* + f^* g)\,dx\right|^2 \le 4 \int_{\mathbb{R}} |f|^2 dx \int_{\mathbb{R}} |g|^2 dx$$

[recall: $|x \cdot y|^2 \le |x|^2 |y|^2$, and in the same way

$$\left|\int f g^*\,dx\right|^2 \le \int |f(x)|^2 dx \int |g(x)|^2 dx$$]

Theorem Define
$$\Delta x = \frac{\int x^2 f f^* dx}{\int f f^* dx}$$
$$\Delta \xi = \frac{\int \xi^2 \hat{f}\, \hat{f}^* d\xi}{\int \hat{f} \hat{f}^* d\xi}$$

Then
$$\Delta x \, \Delta \xi \geq \frac{1}{4\pi}$$

Lemma
$$\int f' f'^* dx = 4\pi^2 \int \xi^2 |\hat{f}|^2 d\xi$$

⌈ proof of Lemma:

use that $\mathcal{F}(Df) = 2\pi i \xi \hat{f}(\xi)$

and that $\int_{\mathbb{R}} |f|^2 dx = \int_{\mathbb{R}} |\hat{f}|^2 d\xi$ ⌋

Proof of theorem

$$(\Delta x)^2 (\Delta \xi)^2 = \frac{\int x^2 f f^* dx}{\int f f^* dx} \; \frac{\int \xi^2 \hat{f} \hat{f}^* d\xi}{\int \hat{f} \hat{f}^* d\xi}$$

$$= \frac{\int x f (xf)^* dx \;\; \int f' f'^* dx}{4\pi^2 \left(\int_{\mathbb{R}} f f^* dx\right)^2}$$

$$\geq \frac{\left(\int_{\mathbb{R}} (x f f'^* + x f^* f')\, dx\right)^2}{16\pi^2 \left(\int |f|^2 dx\right)^2} = \frac{\left(\int x \frac{d}{dx} f f^* \, dx\right)^2}{16\pi^2 \left(\int |f|^2 dx\right)^2}$$

$$= \frac{\left(\int |f|^2 dx\right)^2}{16\pi^2 \int |f|^2 dx} = \frac{1}{16\pi^2}.$$

∎

Filters and signals, filter banks

A discrete signal x is a (double) sequence

$$x = \{x_k\}_{k=-\infty}^{\infty} = (\ldots, x_{-1}, x_0, x_1, \ldots)$$

$$x_k \in \mathbb{R} \quad (\text{or } \mathbb{C}).$$

So x is a function $x : \mathbb{Z} \to \mathbb{C}$ (or $\mathbb{R}$)

Def $\quad x \in \ell^2 \iff \sum_{k=-\infty}^{\infty} |x_k|^2 < \infty$ (finite energy)

Def $\quad$ A filter is an operator, $H : x \mapsto y$, i.e. $y = Hx$ is another signal.

$$x \longrightarrow \boxed{H} \longrightarrow y$$

Def $\quad H$ is linear if $H(\alpha x + \beta y) = \alpha Hx + \beta Hy$

time invariant if $H(Dx) = D H(x)$ for all x

(note Dx is defined by $(Dx)_k = x_{k-1}$)

Def $\quad \delta = \{\delta_k\}_{k=-\infty}^{\infty}, \quad \delta_k = \begin{cases} 1 & \text{if } k=0 \\ 0 & \text{otherwise} \end{cases}$

Def $\quad h = H\delta$ is called the impulse response of the filter

Suppose now that a filter H is Linear Time Invariant (LTI)

We then have $D^n h = H(D^n \delta)$.

Take $x = \{x_k\}_{k=-\infty}^{\infty}$ arbitrary.

Then

$$x = \ldots\ldots\; x_{-2} D^{-2}\delta + x_{-1} D^{-1}\delta + x_0 \delta + x_1 D\delta + x_2 D^2\delta + \ldots$$

$$\begin{aligned}(= \;& \ldots\ldots \\ &+ (0, \ldots\ldots 0, x_{-1}, 0, 0, \ldots) \\ &+ (\ldots\; 0, 0, 0, x_0, 0, 0, 0, \ldots) \\ &+ (\ldots\; 0, 0, 0, 0, x_1, 0, \ldots) \end{aligned}$$

$$= \sum_{n=-\infty}^{\infty} x_n D^n \delta .$$

Hence, if $y = Hx$, we get

$$y = Hx = \sum_{n=-\infty}^{\infty} x_n D^n h = \sum_{n=-\infty}^{\infty} x_n h_{\cdot - n}$$

(i.e., $y = \{y_k\}_{k=-\infty}^{\infty}$,

$$y_k = \sum_{n=-\infty}^{\infty} x_n h_{k-n} = x * h = h * x$$

↑ definition of discrete convolution.

So an LTI - filter is uniquely determined by its impulse respons.

<u>Def</u> A <u>F</u>inite <u>I</u>mpulse <u>R</u>esponse filter (FIR) has only finitely many $h_k \neq 0$.

(otherwise one says IIR - filter)

<u>Def</u> An LTI-filter is <u>causal</u> if $h_k = 0$ for $k < 0$.

About the computer assignment

Some matlab commands (need the manual pages!)

1) convert a wave file to a vector:

[y, Fs, bits] = wavread('fil.wav')

(auread ('fil.au'))

y : data (vector)
Fs : sampling rate (Hz)
bits : number of bits / sample

plot(y) (to see the file)

2) to filter the signal:

yf = filter(b, a, y)

y is the input signal (a vector)
b and a are the vectors that define the filter.

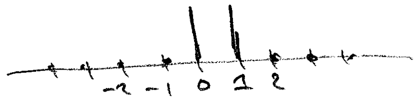
3) You can use simulink to create the filter parameters a and b

- start simulink
  - create new model
    - signal processing library
      - Filter design library
        - Digital Filter Design.
- Double click "filter block"
- choose parameters
- Click "Design Filter"
- File: "export"

Def. (autocorrelation)

$$x \star y = \sum_{n=-\infty}^{\infty} x_{n+\cdot}\, y_n \qquad \left(\text{so} \quad (x \star y)_k = \sum x_{n+k} y_n \right)$$

Example Let $h_k = \begin{cases} \frac{1}{2} & \text{when } k = 0, 1 \\ 0 & \text{otherwise} \end{cases}$



Then $y_k = \frac{1}{2}(x_k + x_{k-1})$.

This is an LTI-filter, that is also causal.

(a low pass filter)

Example Down sampling by 2, $\downarrow 2$

(note the new symbol $\downarrow 2$).

If $x = \{x_k\}_{k=-\infty}^{\infty}$

$$(\downarrow 2)x = (\ldots x_{-4}, x_{-2}, x_0, x_2, x_4, \ldots)$$

Upsampling by 2, $\uparrow 2$

$$(\uparrow 2)x = (x_{-2}, 0, x_{-1}, 0, x_0, 0, x_1, 0, x_2, \ldots)$$

These are linear but not time invariant (why?).

Def The z-transform of a signal x is

(if $x = \{x_k\}_{k=-\infty}^{\infty}$)

$$X(z) = \sum_{k=-\infty}^{\infty} x_k z^{-k} \qquad z \in \mathbb{C}$$

we write

$$x \supset X(z).$$

[Here the argument must be written out to avoid confusion with the DFT, $X(j) = \ldots$]

Example $x_k = \begin{cases} 1 & k \geq 0 \\ 0 & k < 0 \end{cases}$

Then $x \supset X(z)$ with

$$X(z) = \sum_{k=0}^{\infty} z^{-k} = \frac{1}{1-\frac{1}{z}} = \frac{z}{z-1} \qquad (z \in \mathbb{C} \setminus \{1\})$$

The convolution theorem

If $y = h * x$ then $Y(z) = H(z)X(z)$, and vice versa

Proof
$$Y(z) = \sum_{k=-\infty}^{\infty} z^{-k} \sum_{n=-\infty}^{\infty} h_{k-n} x_n$$
$$= \sum_{k=-\infty}^{\infty} z^{-(k-n)} \sum_{n=-\infty}^{\infty} z^{-n} h_{k-n} x_n$$
$$= \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} z^{-k} z^{-n} h_k x_n = H(z)X(z)$$

The Discrete Fourier Transform (of infinite sequences)

$$X(\omega) = \sum_{k=-\infty}^{\infty} x_k e^{-i\omega k}.$$

[NB In this definition $e^{-i\omega k}$, NOT $e^{-2\pi i \omega k}$]

Other common notation:

$$\hat{x}(\omega) = X(\omega).$$

Note that $X(\omega) = X(\underset{\uparrow z}{e^{i\omega}})$

Lemma $X(\omega)$ is 2π-periodic, and

$$x_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{i\omega k} d\omega.$$

Let H be an LTI-filter with impulse response h.

<u>Def</u> $H(\omega)$ is called the <u>frequency response</u>

<u>Example</u> Let $x_t = e^{i\omega t}$, $|\omega| \leq \pi$.

Then $y = Hx$ is given by

$$y_k = \sum_{n=-\infty}^{\infty} h_n x_{k-n} = \sum_{n=-\infty}^{\infty} h_n e^{i\omega(k-n)} = e^{i\omega k} \sum_{n=-\infty}^{\infty} h_n e^{-i\omega n}$$

$$= e^{i\omega k} H(\omega).$$

Write $H(\omega) = |H(\omega)|\, e^{i\phi(\omega)}$ phase function

magnitude

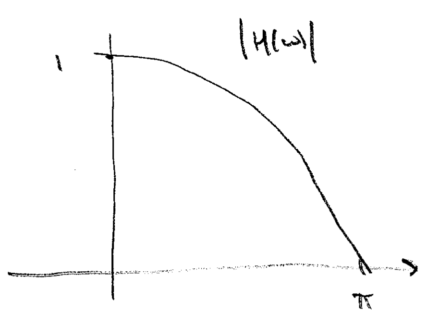
<u>Def</u> If $|H(\omega)| = 1$, H is called an all pass filter.

<u>Example</u> a) $h_0 = h_1 = \frac{1}{2}$, all other $h_k = 0$.

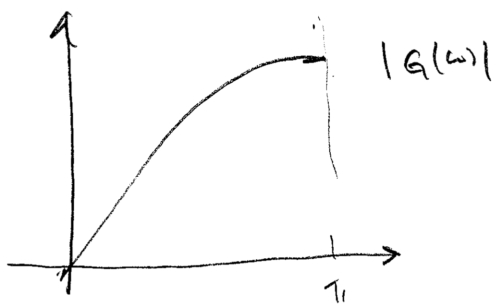
$$\Rightarrow \quad H(\omega) = \frac{1}{2}(1 + e^{-i\omega}) =$$
$$= e^{-i\omega/2} \cos(\omega/2).$$

b) $g_k = \begin{cases} 1/2 & k=0 \\ -1/2 & k=1 \\ 0 & \text{otherwise} \end{cases}$ Then

$$G(\omega) = \frac{1}{2}(1 - e^{-i\omega}) = \dots = i e^{-i\omega/2} \sin \omega/2$$



H is a low pass filter (average)

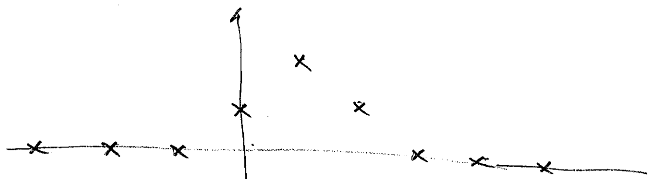


G is a high pass filter. (difference)

Definition
- H has <u>linear phase</u> if $\omega \mapsto \phi(\omega)$ is linear
- H is <u>symmetric</u> (<u>anti symmetric</u>) if $h_k = h_{-k}$ $\quad (h_k = -h_{-k})$
- The <u>group delay</u> of H is

$$\tau(\omega) = -\frac{d\phi}{d\omega}$$

Example Let $h_0 = h_2 = \frac{1}{2}$, $h_1 = 1$ (the others $= 0$)



Then

$$H(\omega) = \frac{1}{2} + e^{-i\omega} + \frac{1}{2} e^{-i2\omega} = e^{-i\omega}\left(1 + \tfrac{1}{2}e^{i\omega} + \tfrac{1}{2}e^{-i\omega}\right)$$

$$= e^{-i\omega} \underbrace{(1 + \cos\omega)}_{|H(\omega)|} \qquad \phi(\omega) = -\omega \quad \ldots \text{linear}$$

Example If H is an LTI-filter and if x is a pure frequency, $x_k = e^{i\omega k}$, then

$$y_k = (h * x)_k = \sum_{n=-\infty}^{\infty} h_n e^{i\omega(k-n)}$$

$$= e^{i\omega k} \sum_{n=-\infty}^{\infty} h_n e^{-i\omega n} = H(\omega) e^{i\omega k}.$$

If $x_k = e^{i\omega_1 k} + e^{i\omega_2 k}$ (sum of two pure frequencies)

$$y_k = H(\omega_1) e^{i\omega_1 k} + H(\omega_2) e^{i\omega_2 k}$$

$$= |H(\omega_1)| e^{i\phi(\omega_1)} e^{i\omega_1 k} + |H(\omega_2)| e^{i\phi(\omega_2)} e^{i\omega_2 k}$$

$$= |H(\omega_1)| e^{-i\omega_1} e^{i\omega_1 k} + |H(\omega_2)| e^{-i\omega_2} e^{i\omega_2 k}$$

$$= |H(\omega_1)| e^{i\omega_1(k-1)} + |H(\omega_2)| e^{i\omega_2(k-1)}$$

Note the shift $k \mapsto (k-1)$.

Filter banks

Def A sequence of signals, $\{\varphi^{(n)}\}_{n=-\infty}^{\infty}$

[note: each $\varphi^{(n)}$ is a sequence, $(\ldots, \varphi^{(n)}_{-2}, \varphi^{(n)}_{-1}, \varphi^{(n)}_{0}, \varphi^{(n)}_{1}, \ldots)$]

is called a basis if every x can be written $x = \sum c_n \varphi^{(n)}$.

But now we need to restrict x.

Def $\|x\| = \sqrt{\sum |x_t|^2}$ (this is sometimes denoted $\|x\|_2$ or $\|x\|_{\ell^2}$)

we say that $x^{(n)} \to x$ in norm if $\|x^{(n)} - x\| \to 0$ when $n \to \infty$.

Def A Hilbert space is a "complete inner product space".

Ex ℓ^2, $\langle x, y\rangle = \sum x_t \bar{y}_t$ (note: complex conjugate on y)

$\|x\| = \sqrt{\langle x, x\rangle}$

Cauchy-Schwarz: $|\langle x, y\rangle| \le \|x\| \, \|y\|$.

A Cauchy sequence is a sequence $(x^{(n)})_{n=1}^{\infty}$ such that $\forall \varepsilon > 0 \ \exists N > 0$ such that $\|x^{(n)} - x^{(m)}\| < \varepsilon$ whenever $n, m > N$.

Def A space is complete if every Cauchy sequence is convergent.

An <u>orthogonal basis</u> $(\varphi^{(n)})$ is a basis that satisfies

$$\langle \varphi^{(j)}, \varphi^{(k)} \rangle = \begin{cases} 1 & \text{if } j=k \\ 0 & \text{otherwise} \end{cases}.$$

We can then write

$$x = \sum_n c_n \varphi^{(n)}, \quad \text{where } c_n = \langle x, \varphi^{(n)} \rangle.$$

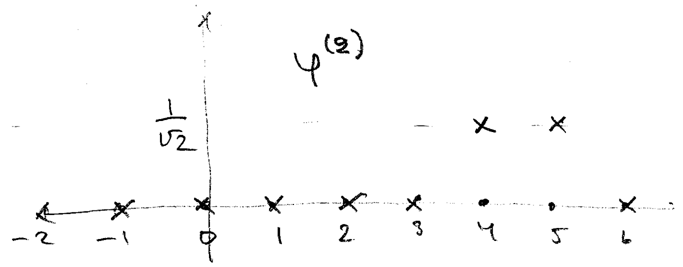
(compare this with the familiar $\mathbb{R}^n$)

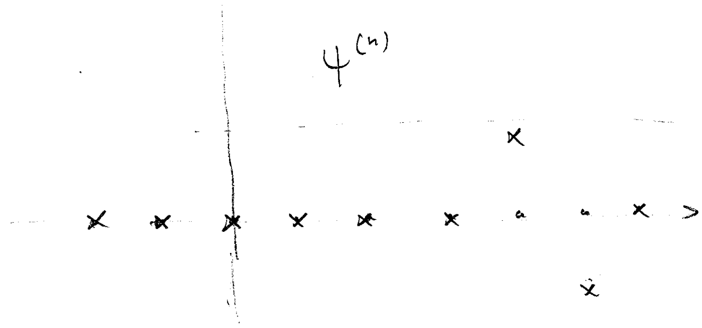
<u>Definition</u> The <u>Haar basis</u> consists of two families of functions,

$$\varphi_k = \begin{cases} 1/\sqrt{2} & \text{when } k=0,1 \\ 0 & \text{otherwise} \end{cases} \qquad \psi_k = \begin{cases} 1/\sqrt{2} & k=0 \\ -1/\sqrt{2} & k=1 \\ 0 & \text{otherwise.} \end{cases}$$

Then let

$$(\varphi^{(2n)})_k = \varphi_{k-2n} \qquad (\varphi^{(2n+1)})_k = \psi_{k-2n}.$$





The coordinates of a sequence $x = (x_k)_{k=-\infty}^{\infty}$ in this basis are

$$y_n^{(0)} := c_{2n} = \langle x, \varphi^{(2n)} \rangle = \frac{1}{\sqrt{2}} (x_{2n} + x_{2n+1}) \qquad (\text{mean})$$

$$y_n^{(1)} := c_{2n+1} = \langle x, \varphi^{(2n+1)} \rangle = \frac{1}{\sqrt{2}} (x_{2n} - x_{2n+1}) \qquad (\text{difference}).$$

These are basis functions in $\ell^2(\mathbb{Z})$ and hence

$$x_k = \sum_n c_n \varphi_k^{(n)} =$$

$$= \sum_n y_n^{(0)} (\varphi^{(2n)})_k + \sum_n y_n^{(1)} (\varphi^{(2n+1)})_k$$

$$= \sum_n y_n^{(0)} \varphi_{k-2n} + \sum_n y_n^{(1)} \psi_{k-2n}.$$

This is almost a convolution, and can be obtained using two filters:

H: $\quad h_k = \varphi_k \qquad h_k^* = \varphi_{-k}$

G: $\quad g_k = \psi_k \qquad g_k^* = \psi_{-k}$

"time reversal"

Hence

$$y_n^{(0)} = \sum_k x_k (\varphi^{(2n)})_k = \sum_k x_k h_{k-2n} = \sum_k x_k h^*_{2n-k}$$

$$= (x * h^*)_{2n}$$

and

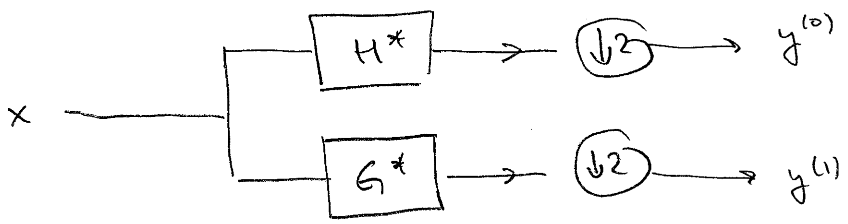
$$y_n^{(1)} = (x * g^*)_{2n}$$

Recall the up- and down sampling:

$(\downarrow 2) x = (\ldots\ x_{-4}, x_{-2}, x_0, x_2, x_4, \ldots)$

$(\uparrow 2) x = (\ldots\ x_{-2}, 0, x_{-1}, 0, x_0, 0, x_1, 0, x_2, \ldots)$

We can obtain the coordinates using two filters:



This procedure is called analysis.

The synthesis is carried out similarly.

$$x_k = \sum_n y_n^{(0)} \varphi_{k-2n} + \sum_n y_n^{(1)} \psi_{k-2n}$$

$$= \sum_n y_n^{(0)} h_{k-2n} + \sum_n y_n^{(1)} g_{k-2n}$$

$$= (v^{(0)} * h)_k + (v^{(1)} * g)_k$$

where $v^{(0)} = (\uparrow 2)\, y^{(0)}$, $v^{(1)} = (\uparrow 2)\, y^{(1)}$

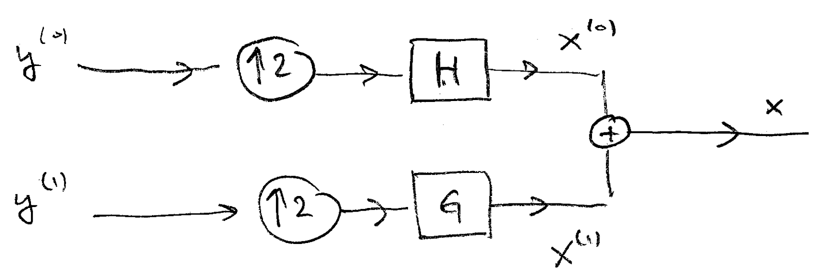
note that

$$\sum_n y_n h_{k-2n} = \sum_n v_{2n} h_{k-2n}$$

$$= \sum_n v_n h_{k-n} = (v * h)_k$$

all odd $v_{2n+1} = 0$

So therefore $\quad x = H\big((\uparrow 2)\, y^{(0)}\big) + G\big((\uparrow 2)\, y^{(1)}\big) = x^{(0)} + x^{(1)}$



and

$$x = \underbrace{\sum \langle x, \varphi^{(2n)} \rangle \varphi^{(2n)}}_{\text{projection onto even}} + \underbrace{\sum \langle x, \varphi^{(2n+1)} \rangle \varphi^{(2n+1)}}_{\text{projection onto odd.}}$$